Chapter 1

Prelude

Mathematics and music are just two of the many ways we respond to the world around us. Mathematics describes quantity and space, while music involves the appreciation of sound. Both math and music are intentional human creations. Like mathematicians, musicians describe concepts like “chords” that cannot be touched or seen. Like musicians, mathematicians speak of “beauty” or “elegance”—in their case, of a particular equation, theorem, or proof. Both disciplines rely on precise notation.

1.1 What is music?

What is the difference between “sound” and “music?” We can probably agree that the styles of popular and classical music that we make or buy are “music.” However, it is difficult to define “music” in general. Some composers, such as John Cage (1912-1992), have deliberately challenged our notions of what music is.

Although there is no universally accepted definition of music, I like the French composer Edgar Varese’s description that “music is organized sound.” This organization can be seen on many different levels, from the sounds that musical instruments make to the way a composer structures a piece. I would add that music exists in time and that music is a form of expression. For the purposes of this class, I’m going to use the definition

Music is the art of organizing sound in time.

Discussion of Chapter 1 videos.

1. The Everyday Ensemble’s “Found Sound Composition” begins with a ticking clock. Other common sounds, such as the bouncing of a ball and a rustling bag, are added in a way that interlocks with the rhythm of the clock. At what point did you realize that the video is “music,” rather than, say, a plot device in a movie? How does organization play a role in our perception of music?

2. In English, musical compositions are often referred to as “pieces,” implying that music comes in finite, contained units. The Ghanian drummers’ music challenges our

expectation that music has a beginning and an end. While their drumming is highly structured, it does not have any particular start or stop.

3. Silence is a part of music—musicians even have symbols for silences, called *rests*. Can silence be music by itself? Watch the video of John Cage's 4'33" (1952). What is the difference between sitting in silence and watching a performance of 4'33"? Or performing the piece yourself? John Cage focused on the listener as the creator of music (or, at least, the person who makes something music rather than sound). He wrote, "If music is the "enjoyment" of "sound", then it must center on not just the side making the sound, but the side listening. In fact, really it is listening that is music. As we savor the sound of rain, music is being created within us" (from "In this time").

4. Is there a difference between poetry and music? Does freestyle rap without backing beats count as music? What about rap in sign language?

5. Algorithmic music is produced when musical notes are chosen by some mathematical set of instructions (*algorithm*). For example, the Online Encyclopedia of Integer Sequences has a "music" feature. It maps numbers to piano keys and plays the notes corresponding to numbers in a mathematical sequence. We listened to sequences A000045 (the Fibonacci numbers 1, 1, 2, 3, 5, 8, 13, ...) and A005843 (the even numbers). Are these sounds "music?" What about sounds made from random sequences of numbers? (In general, music made by some random process is called *aleatory music*.)

6. Does my definition of music as "the art of organizing sound in time" include things that you do not consider "music?"

7. Should our aesthetic judgements—what we like—have any relevance in the definition of music?

8. Are sonic weapons music?

Waveforms. When you record a sound digitally, the picture of the sound that you see is called a *waveform*. It is a visual representation of the vibration of air molecules that produced the sound. The horizontal axis represents time, which is usually measured in seconds. The vertical axis shows you how the air molecules are moving. The rough shape of the waveform when you zoom out gives you information on how the volume of the sound changes. It often looks like a series of blobs. You can also zoom in on the waveform. The patterns that you see when you zoom in give information on the *timbre* (TAM-ber) of the sound. Timbre is difficult to describe, but, loosely, it's the quality of the sound that allows you to distinguish between different instruments, voices, and other sounds.

A *found sound* is a sound that is in the world around you but not intended to be "music" by whoever or whatever made it. Some musicians record found sounds or music produced by other musicians and incorporate them into their own compositions.

Exercise 1.1. Install software that allows you to record and analyze sound. I recommend Audacity for a computer, TwistedWave for iPhone, and MixPad for Android. Audacity has been installed on many computers on campus. Make five-second recordings of yourself talking, humming, clapping, and saying "shh." Comment on the differences between the pattern (or lack of pattern) in their waveforms, both zoomed out and zoomed in.

Solution 1.1. Figure 1.1 shows a sample response. The top image shows the overall shape of the waveforms and gives you information about the rhythm of the sound. Clapping has a repeating pattern representing the rhythm of the claps. Zooming in on the waveform gives information about the "quality" (timbre) of the sound. Only humming shows a repeating pattern. Notice that the speech transitions between a random pattern coinciding with the "s" in music and a repeating pattern coinciding with the "i." In speech, vowels look more like humming than consonants such as "s" do.

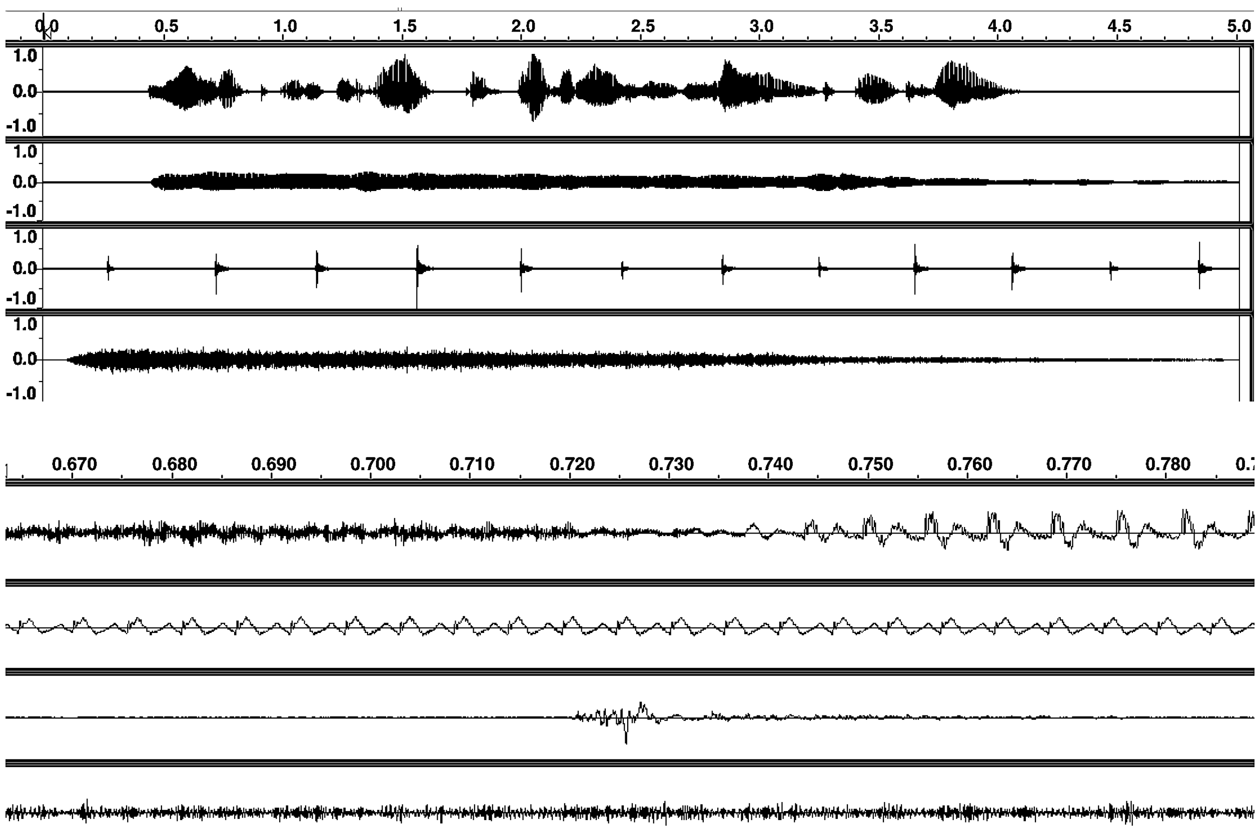

Figure 1.1: *In the top image, the first waveform is me speaking the sentence "Music is the art of organizing sound in time." The others are, in order, humming, clapping, and saying "shh." The bottom image shows what they look like around the 0.7 second mark.*

1.2 What is math?

The philosopher Michael Resnik (1981) called mathematics "a science of pattern." Mathematical science precisely describes structure, both in the physical world and in the abstract. It has been part of a liberal arts education from the beginning. It trains us in using abstraction and in forming logical arguments. Though few people are professional mathematicians, all humans are mathematical thinkers: we engage informally in ideas of quantity, pattern, space, and logic. Music theory, the study of structure in music, is a type of mathematical thinking.

The distinction between *mathematical science* and *mathematical thinking* is in the use of precision and rigor. Mathematical science is precise and logical. It's what you learn in math class and what mathematicians do. Proofs need to be so logically rigorous that they can withstand all potential challenges. In contrast, mathematical thinking is perception or reasoning that involves number, geometry, informal logic, etc. It doesn't have to be formal, and you don't have to have a precise answer. Reading a map and estimating a tip are examples of mathematical thinking, but not mathematical science.

Here are some of the basic terms that mathematicians use to describe what they do.

Definition. A *definition* is a precise description of a mathematical term. Here are a few useful definitions that you should know:

- The *natural numbers* are the numbers $1, 2, 3, 4, \ldots$.
- The *integers* are the numbers $\ldots - 3, -2, -1, 0, 1, 2, 3, \ldots$.
- The *real numbers* are all the numbers on the number line.
- For integers a and b, we say that a *divides* b if a is nonzero and b equals a times some integer k (that is, $b = ak$).
- An *even number* is a number that is divisible by 2. In other words, it equals $2k$, where k is an integer.

Proposition. A *proposition* is a statement that is either true or false. "Two is an even number" and "$3+2=5$" are true propositions. "Two is an odd number" is a false proposition. "Figure out whether two is even," "$5 + x = 2$," and "Is two an even number?" are not propositions. "This sentence is false" sounds like it might be a proposition, but it's neither true nor false. It is an example of a *paradox*.

Axiom. In order to do any useful mathematics, we all have to agree that certain statements, called *axioms*, are true without proof. For example, the *Peano axioms* are needed to establish the definition of integers and the rules of arithmetic. The *Parallel postulate* in geometry states that parallel lines never intersect, where two lines are *parallel* if they are perpendicular to the same line. Surprisingly, it is not possible to use the basic definitions of geometry to prove that parallel lines do not intersect. In fact, the parallel postulate must be accepted as true in Euclidean geometry, which is the geometry of the plane that you learned in high school.[1]

Theorem. A *theorem* is a proposition pertaining to mathematics or logic that has been proven to be true.[2]

[1] Parallel lines *do* intersect on the globe, and this type of geometry is called *spherical geometry*.

[2] The distinction between "theorems" and "true propositions" is not hard-and-fast. Theorems are typically more important, useful, universal, or just more difficult to prove than ordinary true propositions.

Proof. A *proof* is a rigorous, logical argument, written in complete sentences, that demonstrates that a theorem or proposition is true. Proofs build on definitions, axioms, demonstrably true propositions, and other theorems.

The proposition “the sum of two even numbers is an even number” can be proven using a logical argument, assuming that the normal rules of arithmetic are true and following from the definition of an even number. Here is the proof. The box at the end of the paragraph marks the end of the proof.

> Proposition: The sum of any two even numbers is an even number.
>
> *Proof.* Suppose x and y are even numbers. The definition of “even number” means x and y are divisible by 2, so there are integers k and l where $x = 2k$ and $y = 2l$. By the rules of addition and multiplication, $x + y = 2k + 2l = 2(k + l)$, which is an even number because $k + l$ is an integer. □

Notice that this proof implicitly relies on the Peano axioms, because we need the rules of arithmetic and the definition of integers to complete the proof.

Conjecture. A *conjecture* is a proposition pertaining to mathematics or logic that is made without proof. Conjectures are usually “educated guesses” based on some pattern that has been observed. “The sum of two integers is greater than either integer” is a conjecture. Although this conjecture is easily disproven, there exist some famous conjectures that mathematicians have never been able to prove true or false.

Example. An *example* is an illustration of a proposition in a particular instance. Unless you are able to demonstrate that all possible examples of a proposition are true, examples do NOT prove the proposition.

Universal proposition. A *universal proposition* is one that is asserted for infinitely many examples. The universal proposition “the sum of any two even numbers is an even number” is illustrated by examples such as $2 + 4 = 6$ and $-4 + 4 = 0$. However, examples can’t prove that this (or any) universal proposition is true, because there are infinitely many examples, and you can’t test them all.

Counterexample. A *counterexample* is an example that proves that a proposition is false. A counterexample to “The sum of two integers is greater than either integer” is $-1 + 2 = 1$, which is not larger than 2.

Algorithm. An *algorithm* is a set of mathematical instructions. An algorithm for determining whether a number is even is the following: Divide the number by 2. If the remainder is 0, the number is even. If the remainder is not 0, the number is not even.

Exercise 1.2. Prove that 0 is an even number.

Exercise 1.3. A number is *odd* if it equals $2n + 1$, where n is an integer. Suppose that a student wishes to prove that the sum of any two odd numbers is an even number. Explain why the following is not a proof.

> Proposition: The sum of any two odd numbers is an even number.
>
> *Proof.* We can test odd numbers: $1 + 1 = 2$, $1 + 3 = 4$, $3 + 1 = 4$, $1 + 5 = 6$, etc. This also works for negative odd numbers. For example, $-1 + 3 = 2$, $-1 + (-3) = -4$, and $-5 + 5 = 0$. In each case, odd numbers sum to an even number. □

Exercise 1.4. Find a counterexample to the proposition "the product of any two odd numbers is an even number."

Exercise 1.5. Explain why the proposition "All music is made by musical instruments or voices" is false.

Exercise 1.6. Explain why many examples don't prove that a universal proposition is true, but one counterexample *disproves* a universal proposition.

1.3 Solutions to exercises

Solution 1.2. *Proof.* Since $0 = 2 \cdot 0$ and 0 is an integer, 0 is even by the definition of even. □

Solution 1.3. The student has *not* tested all odd numbers. That would be impossible because there are infinitely many odd numbers. A logical argument must be used, such as the proof on page 5.

Solution 1.4. A counterexample is $5 \cdot 3 = 15$, because 5 and 3 are odd and 15 is also odd.

Solution 1.5. You need to find a counterexample—that is, some music that is not made by musical instruments or voices. Two counterexamples are music using found sounds and algorithmic music made by a computer.

Solution 1.6. In the case of a true universal proposition, there are infinitely many examples to test, and testing some of them does not prove that they are all true. Therefore, the rules of logic must be used. In the case of a false universal proposition, demonstrating that it is false in one instance means that the proposition is *not* universal and therefore is false.