Chapter 1

Prelude

Mathematics and music are just two of the many ways we respond to the world around us. Mathematics describes quantity and space, while music involves the appreciation of sound. Both math and music are intentional human creations. Like mathematicians, musicians describe concepts like "chords" that cannot be touched or seen. Like musicians, mathematicians speak of "beauty" or "elegance"—in their case, of a particular equation, theorem, or proof. Both disciplines rely on precise notation.

What is music?

What is the difference between "sound" and "music?" We can probably agree that the styles of popular and classical music that we make or buy are "music." However, it is difficult to define "music" in general. Some composers, such as John Cage (1912-1992), have deliberately challenged our notions of what music is.

Although there is no universally accepted definition of music, I like the French composer Edgar Varese's description that "music is organized sound." This organization can be seen on many different levels, from the sounds that musical instruments make to the way a composer structures a piece. I would add that music exists in time and that music is a form of artistic expression. My favorite definition is

Music is the art of organizing sound in time.

Exercise 1.1. Rank these sounds in order from "most musical" to "least musical." (a) traditional Ghanaian drumming, (b) an ambulance siren, (c) raindrops on a tin roof, (d) the sound of a vacuum cleaner, (e) silence, (f) static, (g) a pop song, (h) birdsong, (i) a dog's bark, (j) freestyle rap, (k) the ticking of a clock, (i) Stockhausen's "Helicopter Quartet." Clearly, there are no right or wrong answers, but pay attention to your reasons for ranking one sound above another. Did you rate the sounds produced by humans higher? Does the presence or absence of a tone you can hum to or a rhythm you can tap to influence your decision? Which distinctions are most difficult to make? Are your judgments influenced by Western cultural norms?

Challenging definitions of music. Throughout history, mathematicians have continually expanded the definitions of fundamental concepts like “number” and “space.” For example, zero wasn’t recognized as a number until the middle ages; the idea of four-dimensional space is relatively recent. Many composers have done the same thing. I would like to start by questioning our ideas about what music is and finding the widest possible “universe” of musical expression. In class, we watched a number of videos that challenge what’s commonly called “music.”

- A *found sound* is a sound that is in the world around you but not intended to be “music” by whoever or whatever made it. Some musicians record found sounds and incorporate them into their own compositions. Music can also be made with *found objects* that are not intended to be musical instruments.

  The Everyday Ensemble’s “Found Sound Composition” uses both found sounds and found objects. It begins with a ticking clock. The musicians play found objects—bouncing balls, rustling bags—to interlock with the rhythm of the clock. At what point did you realize that the clock is “music,” rather than, say, the beginning of a movie? How does organization play a role in our perception of music?
- In English, musical compositions are often referred to as “pieces,” implying that music comes in finite, contained units. The Ghanian drummers’ music challenges the expectation that music has a beginning and an end. In what way is time organized (or not organized) by their drumming?
- Silence is a part of music—musicians even have symbols for silences, called *rests*. Can silence be music by itself? Watch the video of John Cage’s 4’33” (1952).

  Cage focused on the listener as the creator of music. That is, the listener is the artist—the person who makes something music rather than sound. His composition 4’33” throws background sounds into the foreground. He wrote, “If music is the “enjoyment” of “sound”, then it must center on not just the side making the sound, but the side listening. In fact, really it is listening that is music. As we savor the sound of rain, music is being created within us” (from “In this time”). Do you agree? Try picking a natural sound and convince yourself that it is musical.
- Is there a difference between poetry and music? Does freestyle rap without backing beats count as music? What about rap in sign language? Is there a way that dancing can be music?
- Algorithmic music is produced when musical notes are chosen by some mathematical set of instructions (*algorithm*). For example, the Online Encyclopedia of Integer Sequences has a “music” feature. It maps numbers to piano keys and plays the notes corresponding to numbers in a mathematical sequence. We listened to sequences A000045 (the Fibonacci numbers 1, 1, 2, 3, 5, 8, 13, ...) and A005843 (the even numbers). Are these sounds “music?” What about sounds made from random sequences of numbers? (In general, music made by some random process is called *aleatory music*.) Change ringing is an algorithmic method of ringing bells that has been practiced in England for over 500 years. Do you find it musical?
- When rap music first reached a wide audience, some people who didn’t like it complained that it “isn’t music.” Should our aesthetic judgements—what we

like—have any relevance in the definition of music? What about cultural expectations? Does anyone have the "right" to decide what is music?

- Sonic weapons are painfully loud sounds, including tones, recorded songs, and distressing sounds such as crying babies, that are used by the military or police for crowd control or as a form of torture. They may cause permanent hearing damage. Are sonic weapons music?
- Does my definition of music as "the art of organizing sound in time" include things that you do not consider "music?" If so, give examples.

Sound and recording. Sounds are produced by rapid vibrations in the air that are detectable by the ears of humans or other animals. In music, *notes* have volume (loudness or softness) and may have pitch (highness or lowness), while *rests* are silences. Both notes and rests have a beginning (*onset*) and end (*offset*); the difference in time between their onset and offset is called their *duration*.

When you record a sound digitally, the picture of the sound that you see is called a *waveform*. It is a visual representation of the vibration of air molecules that produced the sound. The horizontal axis represents time, which is usually measured in seconds.

The vertical axis shows you *displacement*—how the air molecules are moving back and forth. In reality, molecules are moving in three dimensions, but this two-dimensional picture is good enough in most situations.

Waveforms give us useful information both on a "macro" and "micro" level. The rough shape of the waveform when you zoom out shows how the loudness of the sound changes. It often looks like a series of blobs, with fat blobs for loud sounds; we perceive stronger vibrations as louder. If a definite onset and offset can be detected, you can find the duration of a sound by subtracting the onset from the offset.

You can also zoom in on the waveform. For many musical instruments, the zoomed-in waveform has a pattern that repeats hundreds or thousands of times per second. These patterns give information on the pitch and *timbre* (TAM-ber) of the sound. Timbre is difficult to describe, but, loosely, it's the quality of the sound that allows you to distinguish between different instruments, voices, and other sounds.

Exercise 1.2. Install software that allows you to record and analyze sound. I recommend Audacity for a computer, TwistedWave for iPhone, and MixPad for Android. Audacity has been installed on many computers on campus. Make five-second recordings of yourself talking, humming, clapping, and saying "shh." Comment on the differences between the pattern (or lack of pattern) in their waveforms, both zoomed out and zoomed in.

Solution 1.2. Figure 1.1 shows a sample response. Each sound clip has a duration of 5 seconds. The top image shows the overall shape of the waveforms and gives you information about the rhythm of the sound. We can see that the word "music" has a duration of about a quarter second. Clapping has a repeating pattern representing the rhythm of the claps, which occur roughly at intervals of 0.4 seconds. Zooming in on the waveform gives information about the "quality" (timbre) of the sound. Only humming shows a repeating pattern. Notice that the speech transitions between a random pattern coinciding with the "s" in music and a repeating pattern coinciding with the "i." In speech, vowels look more like humming than consonants such as "s" do.

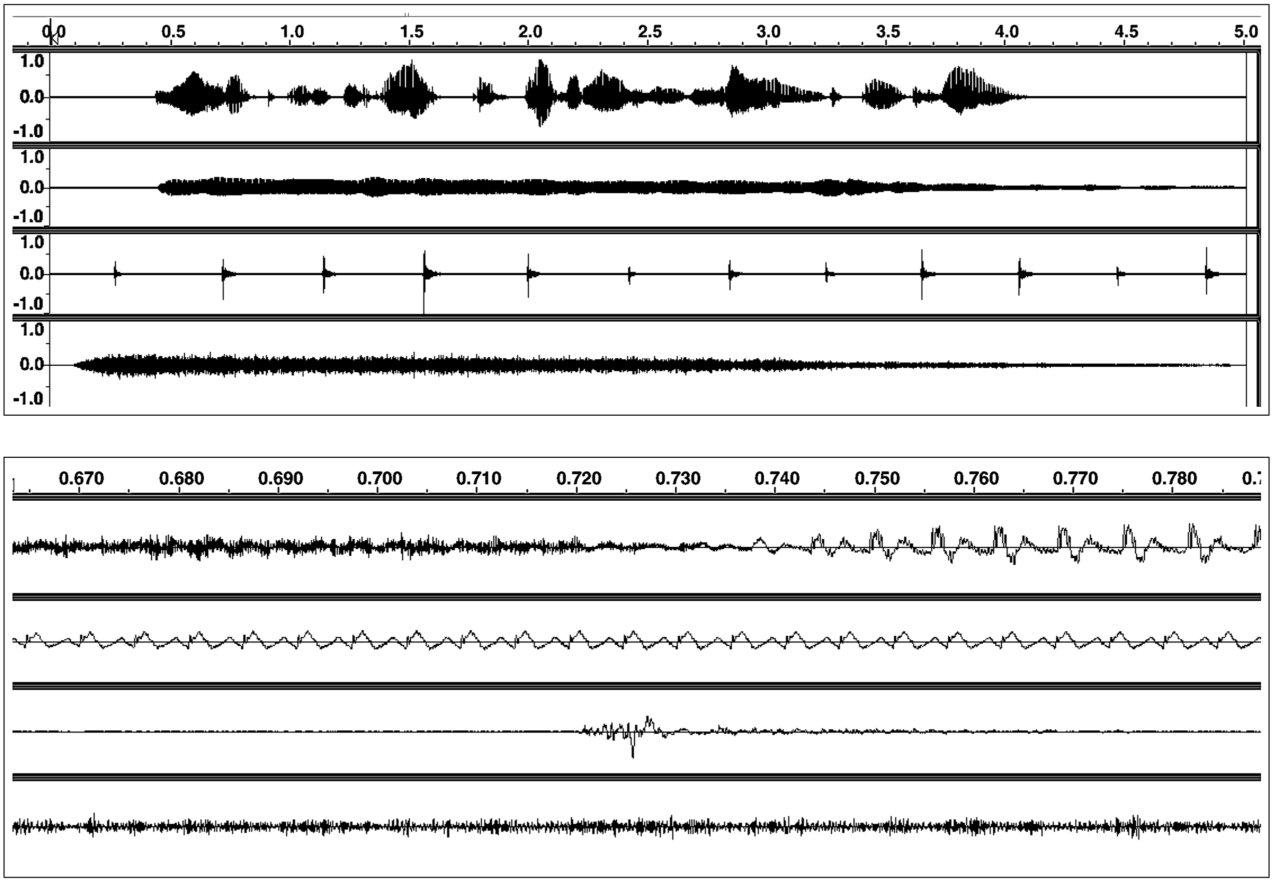

Figure 1.1: *In the top image, the first waveform is me speaking the sentence "Music is the art of organizing sound in time." The others are, in order, humming, clapping, and saying "shh." The bottom image shows what they look like around the 0.7 second mark.*

Music notation. Some musical cultures use notation use *music notation*, which is a symbolic language used to communicate instructions to musicians. There are many different systems of notation used throughout the world. The symbols used in *Western common practice music*—the music often called "Classical"—are probably familiar to you, even if you don't know how to interpret, or "read," them. Figure 1.2 shows an example of a *musical score*. We will learn a few ways of notating music in this class.

What is math?

The philosopher Michael Resnik (1981) called mathematics "a science of pattern." Mathematics precisely describes structure, both in the physical world and in the abstract. It has been part of a liberal arts education from the beginning. It trains us in using abstraction and in forming logical arguments. Though few people are professional mathematicians, all humans are mathematical thinkers: we engage informally in ideas of quantity, pattern, space, and logic. Music theory, the study of structure in music, is a type of mathematical thinking.

The distinction between *mathematical science* and *mathematical thinking* is in the use of precision and rigor. Mathematical science is precise and logical. It's what you learn in

Beethoven

Symphony No. 5

in C Minor

Op. 67

Figure 1.2: *The first page of the score, or set of notated musical instructions, for Beethoven's Fifth Symphony (1808). As in the Audacity screenshot, the horizontal direction represents time, moving from left to right. Each five-line staff is labeled with the name of a different instrument. Symbols with oval "heads" such as 𝅗𝅥 and ♫♩ are notes, while the symbols ▬, 𝄽, and 𝄾 indicate rests. The position of notes in the staff indicates their pitch, with higher notes on top. The instruments play simultaneously, so that a vertical "slice" of the score shows everything that is happening at a particular moment in time. Written music like this is also called sheet music. We'll watch a video that synchronizes the score and music.*

math class and what mathematicians do for a living. Proofs need to be so logically rigorous that they can withstand all potential challenges. In contrast, mathematical thinking is perception or reasoning that involves number, geometry, logical reasoning, etc. It doesn't have to be formal, and you don't have to have a precise answer. Reading a map and estimating a tip are examples of mathematical thinking, but not mathematical science.

How to think like a mathematician

You may be familiar with the puzzle sudoku. The object of the puzzle is to fill a 9×9 grid of squares with the digits 1 through 9 so that each row, each column, and each 3×3 sub-square contains all nine digits. Some of the numbers are filled in already, and you have to fill in the rest.

Sometimes sudoku is advertised "no math involved!" However, there is plenty of math involved—namely, logic, which is a branch of mathematics—although there is no arithmetic. In fact, there is no need to use numbers: any set of nine different symbols would do just as well.

Solving a sudoku puzzle doesn't rise to the level of mathematical science, but it is an example of mathematical thinking. However, many mathematicians have been inspired by the puzzle to ask "scientific" questions. You can even find research papers on the game published in mathematical journals.

Here are some questions about sudoku, roughly organized from the least mathematical to the most mathematical. What does this say about how mathematicians think?

1. What is the solution to this puzzle?
2. How can you create a new sudoku puzzle?
3. Are there variations of sudoku? Let's solve one.
4. Could you make sudoku games of different sizes, like 4×4 grids?
5. How can you change the rules or add new rules to create your own variations on sudoku?
6. Does every puzzle have a solution? If not, create a puzzle with no solution.
7. Do some puzzles have more than one solution? If so, create a puzzle with two solutions.
8. What is a general strategy for solving sudoku puzzles? Prove that your strategy will always work.
9. Which strategy is the "fastest" in that a computer will be able to solve the puzzle in the shortest possible time? Prove that your strategy is the fastest possible.
10. What is a general strategy for creating puzzles? Prove that your strategy results in a puzzle with a unique solution.

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

								1
							2	3
		4			5			
			1					
				3		6		
		7				5	8	
				6	7			
	1				4			
5	2							

Figure 1.3: *Left: a sudoku game to solve. Right: a sudoku game with the minimal number of clues, which has proven to be 17.*

11. How many different grids are solutions to sudoku puzzles? Do some have interesting features? (Look up "automorphic sudoku.")

12. If you exchange all the "1"s and "2"s in a puzzle, or replace the numbers with letters, you get more or less the same puzzle. In what other ways may two solution grids be essentially the same, or "equivalent"? If you don't count these separately, how many different grids are there?

13. Suppose a "transformation" is an action that turns a puzzle into an equivalent puzzle. What is the structure of the set of transformations of all possible puzzles?

14. What is the minimal number of clues that may be given so that a puzzle has a unique solution?

15. How do all these questions generalize to puzzles with $n \times n$ grids, where n is a positive integer?

Exercise 1.3. How do mathematicians think? Describe how the questions changed from the first to last.

Exercise 1.4. Solve the puzzles in Figure 1.3.

Exercise 1.5. Make a sudoku with no solution and make one with multiple solutions.

Exercise 1.6. Find a question that mathematicians have asked about sudoku, or ask one yourself.

Mathematical terminology

Here are some of the basic terms that mathematicians use to describe what they do.

Definition. A *definition* is a precise description of a mathematical term. Here are a few useful definitions that you should know:

- The *natural numbers* are the numbers $1, 2, 3, 4, \ldots$.
- The *integers* are the numbers $\ldots - 3, -2, -1, 0, 1, 2, 3, \ldots$.
- The *real numbers* are all the numbers on the number line.
- For integers a and b, we say that a *divides* b if a is nonzero and b equals a times some integer k (that is, $b = ak$).
- An *even number* is a number that is divisible by 2. In other words, it equals $2k$, where k is an integer.

Proposition. A *proposition* is a statement that is either true or false. "Two is an even number" and "$3+2=5$" are true propositions. "Two is an odd number" is a false proposition. "Figure out whether two is even," "$5 + x = 2$," and "Is two an even number?" are not propositions. "This sentence is false" sounds like it might be a proposition, but it's neither true nor false. It is an example of a *paradox*.

Axiom. In order to do any useful mathematics, we all have to agree that certain statements, called *axioms*, are true without proof. For example, the *Peano axioms* are needed to establish the definition of integers and the rules of arithmetic. The *Parallel postulate* in geometry states that parallel lines never intersect, where two lines are *parallel* if they are perpendicular to the same line. Surprisingly, it is not possible to use the basic definitions of geometry to prove that parallel lines do not intersect. In fact, the parallel postulate must be accepted as true in Euclidean geometry, which is the geometry of the plane that you learned in high school.[1]

Theorem. A *theorem* is a proposition pertaining to mathematics or logic that has been proven to be true.[2]

Proof. A *proof* is a rigorous, logical argument, written in complete sentences, that demonstrates that a theorem or proposition is true. Proofs build on definitions, axioms, demonstrably true propositions, and other theorems.

The proposition "the sum of two even numbers is an even number" can be proven using a logical argument, assuming that the normal rules of arithmetic are true and following from the definition of an even number. Here is the proof. The box at the end of the paragraph marks the end of the proof.

[1] Parallel lines *do* intersect on the globe, and this type of geometry is called *spherical geometry*.

[2] The distinction between "theorems" and "true propositions" is not hard-and-fast. Theorems are typically more important, useful, universal, or just more difficult to prove than ordinary true propositions.

Proposition: The sum of any two even numbers is an even number.

Proof. Suppose x and y are even numbers. The definition of "even number" means x and y are divisible by 2, so there are integers k and l where $x = 2k$ and $y = 2l$. By the rules of addition and multiplication, $x + y = 2k + 2l = 2(k + l)$, which is an even number because $k + l$ is an integer. □

Notice that this proof implicitly relies on the Peano axioms, because we need the rules of arithmetic and the definition of integers to complete the proof.

Conjecture. A *conjecture* is a proposition pertaining to mathematics or logic that is made without proof. Conjectures are usually "educated guesses" based on some pattern that has been observed. "The sum of two integers is greater than either integer" is a conjecture. Although this conjecture is easily disproven, there exist some famous conjectures that mathematicians have never been able to prove true or false.

Example. An *example* is an illustration of a proposition in a particular instance. Unless you are able to demonstrate that all possible examples of a proposition are true, examples do NOT prove the proposition.

Universal proposition. A *universal proposition* is one that is asserted for infinitely many examples. The universal proposition "the sum of any two even numbers is an even number" is illustrated by examples such as $2 + 4 = 6$ and $-4 + 4 = 0$. However, examples can't prove that this (or any) universal proposition is true, because there are infinitely many examples, and you can't test them all.

Counterexample. A *counterexample* is an example that proves that a proposition is false. A counterexample to "The sum of two integers is greater than either integer" is $-1 + 2 = 1$, which is not larger than 2.

Algorithm. An *algorithm* is a set of mathematical instructions. An algorithm for determining whether a number is even is the following: Divide the number by 2. If the remainder is 0, the number is even. If the remainder is not 0, the number is not even.

Exercise 1.7. Prove that 0 is an even number.

Exercise 1.8. A number is *odd* if it equals $2n + 1$, where n is an integer. Suppose that a student wishes to prove that the sum of any two odd numbers is an even number. Explain why the following is not a proof.

Proposition: The sum of any two odd numbers is an even number.

Proof. We can test odd numbers: $1 + 1 = 2$, $1 + 3 = 4$, $3 + 1 = 4$, $1 + 5 = 6$, etc. This also works for negative odd numbers. For example, $-1 + 3 = 2$, $-1 + (-3) = -4$, and $-5 + 5 = 0$. In each case, odd numbers sum to an even number. □

Exercise 1.9. Find a counterexample to the proposition "the product of any two odd numbers is an even number."

Exercise 1.10. Explain why the proposition "All music is made by musical instruments or voices" is false.

Exercise 1.11. Explain why many examples don't prove that a universal proposition is true, but one counterexample *disproves* a universal proposition.

Solutions to exercises

Solution 1.7. *Proof.* Since $0 = 2 \cdot 0$ and 0 is an integer, 0 is even by the definition of even. □

Solution 1.8. The student has *not* tested all odd numbers. That would be impossible because there are infinitely many odd numbers. A logical argument must be used, such as the proof on page 1–9.

Solution 1.9. A counterexample is $5 \cdot 3 = 15$, because 5 and 3 are odd and 15 is also odd.

Solution 1.10. You need to find a counterexample—that is, some music that is not made by musical instruments or voices. Two counterexamples are music using found sounds and algorithmic music made by a computer.

Solution 1.11. In the case of a true universal proposition, there are infinitely many examples to test, and testing some of them does not prove that they are all true. Therefore, the rules of logic must be used. In the case of a false universal proposition, demonstrating that it is false in one instance means that the proposition is *not* universal and therefore is false.