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Chapter 1

Prelude

Mathematics and music are just two of the many ways we respond to the world around us. Mathematics describes quantity and space, while music involves the appreciation of sound. Both math and music are intentional human creations. Like mathematicians, musicians describe concepts like “chords” that cannot be touched or seen. Like musicians, mathematicians speak of “beauty” or “elegance”—in their case, of a particular equation, theorem, or proof. Both disciplines rely on precise notation.

What is music?

What is the difference between “sound” and “music?” We can probably agree that the styles of popular and classical music that we make or buy are “music.” However, it is difficult to define “music” in general. Some composers, such as John Cage (1912-1992), have deliberately challenged our notions of what music is.

Although there is no universally accepted definition of music, I like the French composer Edgar Varese’s description that “music is organized sound.” This organization can be seen on many different levels, from the sounds that musical instruments make to the way a composer structures a piece. I would add that music exists in time and that music is a form of artistic expression. My favorite definition is

Music is the art of organizing sound in time.

Exercise 1.1. Rank these sounds in order from “most musical” to “least musical.” (a) traditional Ghanaian drumming, (b) an ambulance siren, (c) raindrops on a tin roof, (d) the sound of a vacuum cleaner, (e) silence, (f) static, (g) a pop song, (h) birdsong, (i) a dog’s bark, (j) freestyle rap, (k) the ticking of a clock, (i) Stockhausen’s “Helicopter Quartet.” Clearly, there are no right or wrong answers, but pay attention to your reasons for ranking one sound above another. Did you rate the sounds produced by humans higher? Does the presence or absence of a tone you can hum to or a rhythm you can tap to influence your decision? Which distinctions are most difficult to make? Are your judgments influenced by Western cultural norms?

Challenging definitions of music. Throughout history, mathematicians have continually expanded the definitions of fundamental concepts like "number" and "space." For example, zero wasn't recognized as a number until the middle ages; the idea of four-dimensional space is relatively recent. Many composers have done the same thing. I would like to start by questioning our ideas about what music is and finding the widest possible "universe" of musical expression. In class, we watched a number of videos that challenge what's commonly called "music."

- A *found sound* is a sound that is in the world around you but not intended to be "music" by whoever or whatever made it. Some musicians record found sounds and incorporate them into their own compositions. Music can also be made with *found objects* that are not intended to be musical instruments.

  The Everyday Ensemble's "Found Sound Composition" uses both found sounds and found objects. It begins with a ticking clock. The musicians play found objects—bouncing balls, rustling bags—to interlock with the rhythm of the clock. At what point did you realize that the clock is "music," rather than, say, the beginning of a movie? How does organization play a role in our perception of music?
- In English, musical compositions are often referred to as "pieces," implying that music comes in finite, contained units. The Ghanian drummers' music challenges the expectation that music has a beginning and an end. In what way is time organized (or not organized) by their drumming?
- Silence is a part of music—musicians even have symbols for silences, called *rests*. Can silence be music by itself? Watch the video of John Cage's 4'33" (1952).

  Cage focused on the listener as the creator of music. That is, the listener is the artist—the person who makes something music rather than sound. His composition 4'33" throws background sounds into the foreground. He wrote, "If music is the "enjoyment" of "sound", then it must center on not just the side making the sound, but the side listening. In fact, really it is listening that is music. As we savor the sound of rain, music is being created within us" (from "In this time"). Do you agree? Try picking a natural sound and convince yourself that it is musical.
- Is there a difference between poetry and music? Does freestyle rap without backing beats count as music? What about rap in sign language? Is there a way that dancing can be music?
- Algorithmic music is produced when musical notes are chosen by some mathematical set of instructions (*algorithm*). For example, the Online Encyclopedia of Integer Sequences has a "music" feature. It maps numbers to piano keys and plays the notes corresponding to numbers in a mathematical sequence. We listened to sequences A000045 (the Fibonacci numbers 1, 1, 2, 3, 5, 8, 13, …) and A005843 (the even numbers). Are these sounds "music?" What about sounds made from random sequences of numbers? (In general, music made by some random process is called *aleatory music*.) Change ringing is an algorithmic method of ringing bells that has been practiced in England for over 500 years. Do you find it musical?
- When rap music first reached a wide audience, some people who didn't like it complained that it "isn't music." Should our aesthetic judgements—what we

like—have any relevance in the definition of music? What about cultural expectations? Does anyone have the "right" to decide what is music?

- Sonic weapons are painfully loud sounds, including tones, recorded songs, and distressing sounds such as crying babies, that are used by the military or police for crowd control or as a form of torture. They may cause permanent hearing damage. Are sonic weapons music?
- Does my definition of music as "the art of organizing sound in time" include things that you do not consider "music?" If so, give examples.

Sound and recording. Sounds are produced by rapid vibrations in the air that are detectable by the ears of humans or other animals. In music, *notes* have volume (loudness or softness) and may have pitch (highness or lowness), while *rests* are silences. Both notes and rests have a beginning (*onset*) and end (*offset*); the difference in time between their onset and offset is called their *duration*.

When you record a sound digitally, the picture of the sound that you see is called a *waveform*. It is a visual representation of the vibration of air molecules that produced the sound. The horizontal axis represents time, which is usually measured in seconds.

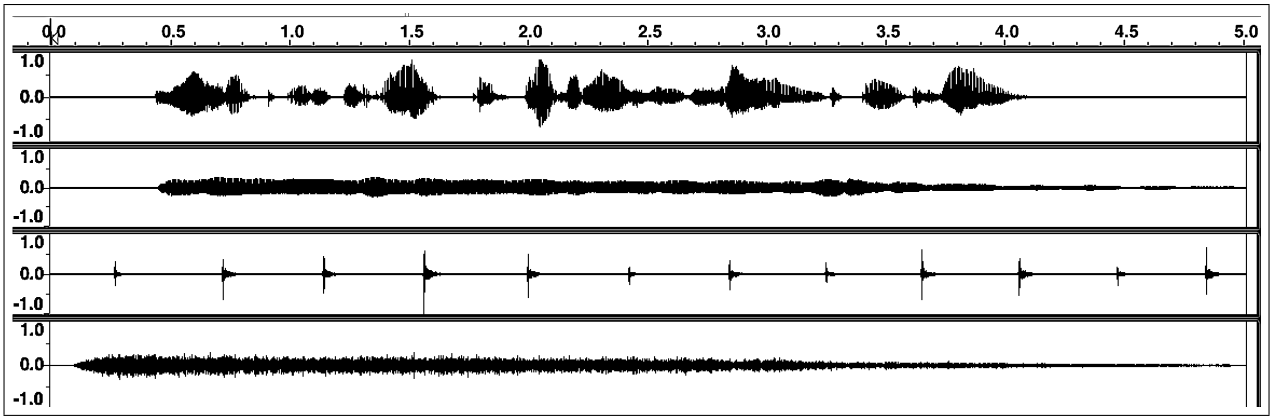
The vertical axis shows you *displacement*—how the air molecules are moving back and forth. In reality, molecules are moving in three dimensions, but this two-dimensional picture is good enough in most situations.

Waveforms give us useful information both on a "macro" and "micro" level. The rough shape of the waveform when you zoom out shows how the loudness of the sound changes. It often looks like a series of blobs, with fat blobs for loud sounds; we perceive stronger vibrations as louder. If a definite onset and offset can be detected, you can find the duration of a sound by subtracting the onset from the offset.

You can also zoom in on the waveform. For many musical instruments, the zoomed-in waveform has a pattern that repeats hundreds or thousands of times per second. These patterns give information on the pitch and *timbre* (TAM-ber) of the sound. Timbre is difficult to describe, but, loosely, it's the quality of the sound that allows you to distinguish between different instruments, voices, and other sounds.

Exercise 1.2. Install software that allows you to record and analyze sound. I recommend Audacity for a computer, TwistedWave for iPhone, and MixPad for Android. Audacity has been installed on many computers on campus. Make five-second recordings of yourself talking, humming, clapping, and saying "shh." Comment on the differences between the pattern (or lack of pattern) in their waveforms, both zoomed out and zoomed in.

Solution 1.2. Figure 1.1 shows a sample response. Each sound clip has a duration of 5 seconds. The top image shows the overall shape of the waveforms and gives you information about the rhythm of the sound. We can see that the word "music" has a duration of about a quarter second. Clapping has a repeating pattern representing the rhythm of the claps, which occur roughly at intervals of 0.4 seconds. Zooming in on the waveform gives information about the "quality" (timbre) of the sound. Only humming shows a repeating pattern. Notice that the speech transitions between a random pattern coinciding with the "s" in music and a repeating pattern coinciding with the "i." In speech, vowels look more like humming than consonants such as "s" do.



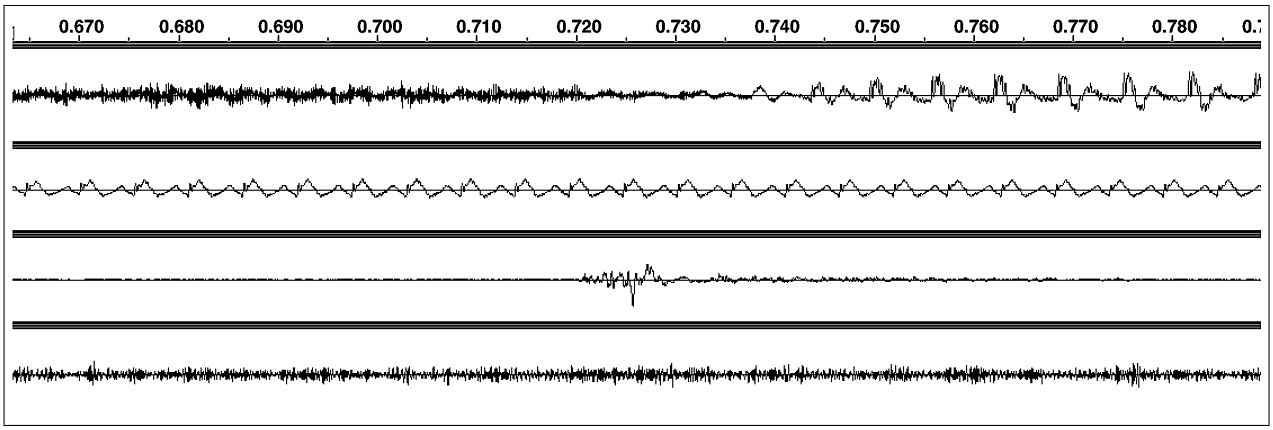


Figure 1.1: *In the top image, the first waveform is me speaking the sentence "Music is the art of organizing sound in time." The others are, in order, humming, clapping, and saying "shh." The bottom image shows what they look like around the 0.7 second mark.*

Music notation. Some musical cultures use notation use *music notation*, which is a symbolic language used to communicate instructions to musicians. There are many different systems of notation used throughout the world. The symbols used in *Western common practice music*—the music often called "Classical"—are probably familiar to you, even if you don't know how to interpret, or "read," them. Figure 1.2 shows an example of a *musical score*. We will learn a few ways of notating music in this class.

What is math?

The philosopher Michael Resnik (1981) called mathematics "a science of pattern." Mathematics precisely describes structure, both in the physical world and in the abstract. It has been part of a liberal arts education from the beginning. It trains us in using abstraction and in forming logical arguments. Though few people are professional mathematicians, all humans are mathematical thinkers: we engage informally in ideas of quantity, pattern, space, and logic. Music theory, the study of structure in music, is a type of mathematical thinking.

The distinction between *mathematical science* and *mathematical thinking* is in the use of precision and rigor. Mathematical science is precise and logical. It's what you learn in

Beethoven

Symphony No. 5

in C Minor

Op. 67

Figure 1.2: *The first page of the score, or set of notated musical instructions, for Beethoven's Fifth Symphony (1808). As in the Audacity screenshot, the horizontal direction represents time, moving from left to right. Each five-line staff is labeled with the name of a different instrument. Symbols with oval "heads" such as 𝅗𝅥 and ♫♩ are notes, while the symbols 𝄼, 𝄽, and 𝄾 indicate rests. The position of notes in the staff indicates their pitch, with higher notes on top. The instruments play simultaneously, so that a vertical "slice" of the score shows everything that is happening at a particular moment in time. Written music like this is also called sheet music. We'll watch a video that synchronizes the score and music.*

math class and what mathematicians do for a living. Proofs need to be so logically rigorous that they can withstand all potential challenges. In contrast, mathematical thinking is perception or reasoning that involves number, geometry, logical reasoning, etc. It doesn't have to be formal, and you don't have to have a precise answer. Reading a map and estimating a tip are examples of mathematical thinking, but not mathematical science.

How to think like a mathematician

You may be familiar with the puzzle sudoku. The object of the puzzle is to fill a 9×9 grid of squares with the digits 1 through 9 so that each row, each column, and each 3×3 sub-square contains all nine digits. Some of the numbers are filled in already, and you have to fill in the rest.

Sometimes sudoku is advertised "no math involved!" However, there is plenty of math involved—namely, logic, which is a branch of mathematics—although there is no arithmetic. In fact, there is no need to use numbers: any set of nine different symbols would do just as well.

Solving a sudoku puzzle doesn't rise to the level of mathematical science, but it is an example of mathematical thinking. However, many mathematicians have been inspired by the puzzle to ask "scientific" questions. You can even find research papers on the game published in mathematical journals.

Here are some questions about sudoku, roughly organized from the least mathematical to the most mathematical. What does this say about how mathematicians think?

1. What is the solution to this puzzle?
2. How can you create a new sudoku puzzle?
3. Are there variations of sudoku? Let's solve one.
4. Could you make sudoku games of different sizes, like 4×4 grids?
5. How can you change the rules or add new rules to create your own variations on sudoku?
6. Does every puzzle have a solution? If not, create a puzzle with no solution.
7. Do some puzzles have more than one solution? If so, create a puzzle with two solutions.
8. What is a general strategy for solving sudoku puzzles? Prove that your strategy will always work.
9. Which strategy is the "fastest" in that a computer will be able to solve the puzzle in the shortest possible time? Prove that your strategy is the fastest possible.
10. What is a general strategy for creating puzzles? Prove that your strategy results in a puzzle with a unique solution.

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

								1
							2	3
		4			5			
			1					
				3		6		
		7				5	8	
				6	7			
	1				4			
5	2							

Figure 1.3: *Left: a sudoku game to solve. Right: a sudoku game with the minimal number of clues, which has proven to be 17.*

11. How many different grids are solutions to sudoku puzzles? Do some have interesting features? (Look up "automorphic sudoku.")

12. If you exchange all the "1"s and "2"s in a puzzle, or replace the numbers with letters, you get more or less the same puzzle. In what other ways may two solution grids be essentially the same, or "equivalent"? If you don't count these separately, how many different grids are there?

13. Suppose a "transformation" is an action that turns a puzzle into an equivalent puzzle. What is the structure of the set of transformations of all possible puzzles?

14. What is the minimal number of clues that may be given so that a puzzle has a unique solution?

15. How do all these questions generalize to puzzles with $n \times n$ grids, where n is a positive integer?

Exercise 1.3. How do mathematicians think? Describe how the questions changed from the first to last.

Exercise 1.4. Solve the puzzles in Figure 1.3.

Exercise 1.5. Make a sudoku with no solution and make one with multiple solutions.

Exercise 1.6. Find a question that mathematicians have asked about sudoku, or ask one yourself.

Mathematical terminology

Here are some of the basic terms that mathematicians use to describe what they do.

Definition. A *definition* is a precise description of a mathematical term. Here are a few useful definitions that you should know:

- The *natural numbers* are the numbers $1, 2, 3, 4, \ldots$.
- The *integers* are the numbers $\ldots - 3, -2, -1, 0, 1, 2, 3, \ldots$.
- The *real numbers* are all the numbers on the number line.
- For integers a and b, we say that a *divides* b if a is nonzero and b equals a times some integer k (that is, $b = ak$).
- An *even number* is a number that is divisible by 2. In other words, it equals $2k$, where k is an integer.

Proposition. A *proposition* is a statement that is either true or false. "Two is an even number" and "$3+2 = 5$" are true propositions. "Two is an odd number" is a false proposition. "Figure out whether two is even," "$5 + x = 2$," and "Is two an even number?" are not propositions. "This sentence is false" sounds like it might be a proposition, but it's neither true nor false. It is an example of a *paradox*.

Axiom. In order to do any useful mathematics, we all have to agree that certain statements, called *axioms*, are true without proof. For example, the *Peano axioms* are needed to establish the definition of integers and the rules of arithmetic. The *Parallel postulate* in geometry states that parallel lines never intersect, where two lines are *parallel* if they are perpendicular to the same line. Surprisingly, it is not possible to use the basic definitions of geometry to prove that parallel lines do not intersect. In fact, the parallel postulate must be accepted as true in Euclidean geometry, which is the geometry of the plane that you learned in high school.[1]

Theorem. A *theorem* is a proposition pertaining to mathematics or logic that has been proven to be true.[2]

Proof. A *proof* is a rigorous, logical argument, written in complete sentences, that demonstrates that a theorem or proposition is true. Proofs build on definitions, axioms, demonstrably true propositions, and other theorems.

The proposition "the sum of two even numbers is an even number" can be proven using a logical argument, assuming that the normal rules of arithmetic are true and following from the definition of an even number. Here is the proof. The box at the end of the paragraph marks the end of the proof.

[1] Parallel lines *do* intersect on the globe, and this type of geometry is called *spherical geometry*.

[2] The distinction between "theorems" and "true propositions" is not hard-and-fast. Theorems are typically more important, useful, universal, or just more difficult to prove than ordinary true propositions.

Proposition: The sum of any two even numbers is an even number.

Proof. Suppose x and y are even numbers. The definition of "even number" means x and y are divisible by 2, so there are integers k and l where $x = 2k$ and $y = 2l$. By the rules of addition and multiplication, $x + y = 2k + 2l = 2(k + l)$, which is an even number because $k + l$ is an integer. □

Notice that this proof implicitly relies on the Peano axioms, because we need the rules of arithmetic and the definition of integers to complete the proof.

Conjecture. A *conjecture* is a proposition pertaining to mathematics or logic that is made without proof. Conjectures are usually "educated guesses" based on some pattern that has been observed. "The sum of two integers is greater than either integer" is a conjecture. Although this conjecture is easily disproven, there exist some famous conjectures that mathematicians have never been able to prove true or false.

Example. An *example* is an illustration of a proposition in a particular instance. Unless you are able to demonstrate that all possible examples of a proposition are true, examples do NOT prove the proposition.

Universal proposition. A *universal proposition* is one that is asserted for infinitely many examples. The universal proposition "the sum of any two even numbers is an even number" is illustrated by examples such as $2 + 4 = 6$ and $-4 + 4 = 0$. However, examples can't prove that this (or any) universal proposition is true, because there are infinitely many examples, and you can't test them all.

Counterexample. A *counterexample* is an example that proves that a proposition is false. A counterexample to "The sum of two integers is greater than either integer" is $-1 + 2 = 1$, which is not larger than 2.

Algorithm. An *algorithm* is a set of mathematical instructions. An algorithm for determining whether a number is even is the following: Divide the number by 2. If the remainder is 0, the number is even. If the remainder is not 0, the number is not even.

Exercise 1.7. Prove that 0 is an even number.

Exercise 1.8. A number is *odd* if it equals $2n + 1$, where n is an integer. Suppose that a student wishes to prove that the sum of any two odd numbers is an even number. Explain why the following is not a proof.

Proposition: The sum of any two odd numbers is an even number.

Proof. We can test odd numbers: $1 + 1 = 2$, $1 + 3 = 4$, $3 + 1 = 4$, $1 + 5 = 6$, etc. This also works for negative odd numbers. For example, $-1 + 3 = 2$, $-1 + (-3) = -4$, and $-5 + 5 = 0$. In each case, odd numbers sum to an even number. □

Exercise 1.9. Find a counterexample to the proposition "the product of any two odd numbers is an even number."

Exercise 1.10. Explain why the proposition "All music is made by musical instruments or voices" is false.

Exercise 1.11. Explain why many examples don't prove that a universal proposition is true, but one counterexample *disproves* a universal proposition.

Solutions to exercises

Solution 1.7. *Proof.* Since $0 = 2 \cdot 0$ and 0 is an integer, 0 is even by the definition of even. □

Solution 1.8. The student has *not* tested all odd numbers. That would be impossible because there are infinitely many odd numbers. A logical argument must be used, such as the proof on page 1–9.

Solution 1.9. A counterexample is $5 \cdot 3 = 15$, because 5 and 3 are odd and 15 is also odd.

Solution 1.10. You need to find a counterexample—that is, some music that is not made by musical instruments or voices. Two counterexamples are music using found sounds and algorithmic music made by a computer.

Solution 1.11. In the case of a true universal proposition, there are infinitely many examples to test, and testing some of them does not prove that they are all true. Therefore, the rules of logic must be used. In the case of a false universal proposition, demonstrating that it is false in one instance means that the proposition is *not* universal and therefore is false.

Chapter 2

Math for Poets

> . . .
> But most by Numbers judge a Poet's Song,
> And smooth or rough, with them, is right or wrong;
> . . .
> These Equal Syllables alone require,
> Tho' oft the Ear the open Vowels tire,
> While Expletives their feeble Aid do join,
> And ten low Words oft creep in one dull Line,
> . . .
>
> —Alexander Pope, *An Essay on Criticism* (1709)

Numerical patterns have fascinated humans for millennia: numbers that are powers of other numbers, squares that are sums of squares, numbers that form intriguing lists. This is the story of one of the earliest studies of rhythm, an investigation that led ancient Indian scholars to discover the mathematical patterns that Westerners know as the Fibonacci numbers, Pascal's triangle, and the binary counting system. Although our story initially concerns rhythm in poetry, the Ancient Indians' ability and fascination with exploring rhythmic patterns also had a profound influence on their music.

Meter as binary pattern

In English, a poetic rhythm, called a *meter*, is a pattern of stressed and unstressed syllables. English poets use about a dozen different meters. Much poetry, including Shakespeare's plays, is written in *iambic pentameter*—five pairs of alternating unstressed and stressed syllables to a line. Alexander Pope's 700+ line iambic pentameter poem *An Essay on Criticism* (1709) is a good example. In the excerpt that begins this chapter, he ridicules critics who judge poetry "by numbers"—that is, solely on how well a poet follows strict metrical rules.

While English poets use relatively few meters, there are hundreds of different meters in Sanskrit, the classical language of India. Syllables in Sanskrit poetry are classified by duration (short or long) rather than stress. Any Sanskrit meter can be written as a *binary pattern*—a pattern of any length formed by two symbols. For example, there are eight binary patterns of length 3 that are formed from the letters L and S: LLL, SLL, LSL, SSL, LLS, SLS, LSS, and SSS. These correspond with the eight three-syllable meters, using S for a short syllable and L for a long syllable.

Pingala is thought to be the first Indian scholar to study meter mathematically. He probably lived in the last few centuries BC. As is typical in ancient Indian literature, Pingala's writings took the form of short, cryptic verses, or *sūtras*, which served as memory aids for a larger set of concepts passed on orally. We are dependent on medieval commentators for transmission and interpretation of Pingala's writings.

Here are two of the questions that Pingala solved:

1. What is a reliable way to list all the meters with a given number of syllables?
2. How many meters have a given number of syllables?

Problem 1: listing meters. There are a number of ways to solve this problem. Pingala's solution would result in the one-syllable meters listed as

L
S

the two-syllable meters being listed as

LL
SL
LS
SS

and the three-syllable meters like this:

LLL
SLL
LSL
SSL
LLS
SLS
LSS
SSS

Here is how the four-syllable patterns would be listed:

LLLL
SLLL
LSLL
SSLL
LLSL
SLSL
LSSL
SSSL
LLLS
SLLS
LSLS
SSLS
LLSS
SLSS
LSSS
SSSS

Several patterns are observable in these lists. The first column alternates L and S, the second alternates pairs of L's and pairs of S's, the third alternates four copies of the letters, and so on. There is symmetry in the lists, in the sense that the first pattern is equivalent to the last with the letters exchanged, and this is true for each pair of patterns that are at the same distance from the beginning and end. In addition, there is a relationship between successive lists: for example, the list of four-syllable meters is formed from the list of three-syllable meters by first adding L's to the end of the list, then adding S's. This last observation is useful for describing an algorithm that will produce all meters of length n in the order that Pingala did.

Exercise 2.1. Write the list of five-syllable meters.

Theorem 2.1 (Listing n-syllable meters). *The list of one-syllable patterns is {L, S}. Suppose a list of n-syllable patterns is formed from a nonrepeating list of all $(n-1)$-syllable patterns by adding L's to the end of each $(n-1)$-syllable pattern, followed by the list resulting from adding S's to the end of each $(n-1)$-syllable pattern. Then each n-syllable pattern will occur exactly once in the new list.*

Proof. It is clear that each of the one-syllable meters ($n = 1$) is listed once. Suppose that the algorithm results in each of the $(n-1)$-syllable patterns being listed exactly once. Use the algorithm to form a list of n-syllable patterns. Choose any n-syllable pattern. We want to show that your chosen pattern occurs exactly once in the new list. If the pattern ends in an L, then the algorithm shows that it appears exactly once in the first half of the list, because the pattern of its first $(n-1)$ syllables appear exactly once in the list of $(n-1)$-syllable patterns. If it ends in an S, it appears exactly once in the second half of the list for the same reason. □

The proof essentially says that if the one-syllable patterns are correct, then the two-syllable patterns are correct, then the three-syllable patterns are correct, and so on, until your chosen length is reached—sort of like a row of dominoes falling down. This type of reasoning is called *proof by induction.* Although it may seem reasonable to argue this way, in fact, the *axiom of induction* is required to allow it.

Problem 2: counting meters. How many meters have n syllables? Counting the patterns on the lists I have typed, you see the numbers 2, 4, and 8, which are equal to 2^1, 2^2, and 2^3. You might conjecture that there are 16 (or 2^4) four-syllable meters, and, in general, there are 2^n n-syllable meters. This is correct. It follows so closely from the theorem that mathematicians would called it a *corollary*, which is a theorem that may be proven from another theorem without much effort.

Corollary 2.1 (Counting n-syllable meters). *The number of n-syllable meters is 2^n.*

Exercise 2.2. How long is the list on which LLSSLSL appears?

Exercise 2.3. Assuming that the theorem has been proven true, explain why the corollary is true. You don't need to write a formal proof, but use complete sentences.

The binary number system Since there's nothing special about the letters L and S, the previous theorem and its corollary generalize to any set of binary patterns. For example, there are 2^5 patterns of length 5 that are formed from the letters a and b. In some ways, Pingala anticipated the development of the binary number system. The binary number system is a base-two positional number system (our number system is a base-ten positional system). It has two digits, 0 and 1, and its place values are powers of two—therefore, every number is also a binary pattern. The decimal numbers 1, 2, 8, and 11 have binary

1 beat	2 beats	3 beats	4 beats	5 beats
S	L SS	SL LS	LL SLS	SLL LLS
		SSS	SSL LSS	LSL SSLS
			SSSS	SSSL SLSS
				LSSS
				SSSSS

Figure 2.1: *Meters listed by duration*

representation 1, 10, 1000, and 1011, respectively. The binary number system was not fully described until Gottfried Leibniz did so in the seventeenth century.

Exercise 2.4. Suppose you flip a coin three times and write down the binary pattern of heads and tails, using H and T. The order of flips makes a difference—that is HHT is different from HTH. How many patterns of three coin flips are there? List them and use your list to compute the likelihood of getting tails exactly once if you flip three times. Describe how you would list and count the patterns for any number of coin flips.

Exercise 2.5. Here are the binary numbers from 8 (1000 in binary) to 15 (1111 in binary):

1000
1001
1010
1011
1100
1101
1110
1111

How is this sequence related to the list of three-syllable meters? Conjecture how many binary numbers have five digits and list them. (Extra Credit: Learn more about the binary number system and determine whether your answer is correct.)

The Hemachandra-Fibonacci numbers

The 12th-century writer Ācārya Hemachandra also studied poetic meter. A *mora* is the durational unit of Sanskrit poetry; short syllables count as one mora and long syllables two morae, which we'll call "beats." Instead of counting meters with a fixed number of syllables, Hemachandra counted meters having a fixed duration. For example, here are the three meters of three beats: SL, LS, and SSS. More meters are listed in Figure 2.1.

Exercise 2.6. Before you go on, count the number of meters for duration one through five and make a conjecture about the number of meters with six beats and the formula for finding the number of meters with any arbitrary number of beats. Try the worksheet for a more thorough exploration of the problem.

Hemachandra noticed that each number in the sequence is the sum of the two previous numbers. Since the first two numbers are 1 and 2, the numbers form the sequence

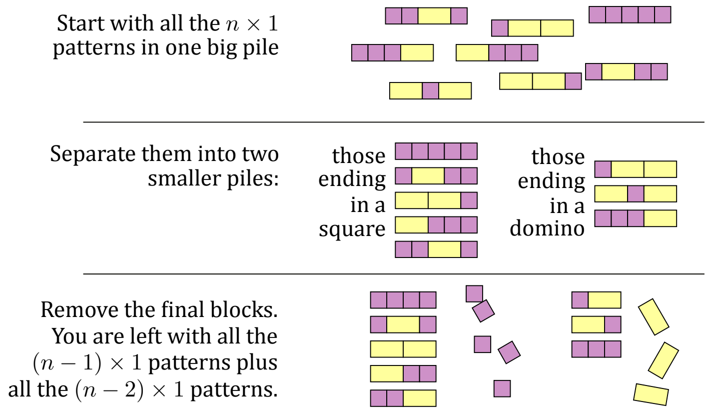


Figure 2.2: *Hemachandra's problem is equivalent to the "domino-square problem": in how many ways can* 1×2 *dominoes and* 1×1 *squares tile a* $1 \times n$ *rectangle? Here is a visual demonstration that the* n*th number in the sequence is the sum of the two preceding numbers.*

$1, 2, 3, 5, 8, 13 \ldots$. In other words, he discovered the "Fibonacci" numbers—about fifty years before Fibonacci did.[1] Indian poets and drummers know these numbers as "Hemachandra numbers."

Theorem 2.2. *The sequence of numbers of meters with* n *beats, beginning with* $n = 1$*, is the Hemachandra sequence,* $1, 2, 3, 5, 8, 13, \ldots$. *When* $n > 2$*, each number in the sequence is the sum of the two previous numbers.*

Proof. Suppose $H[n]$ represents the nth number in the sequence, which equals the total number of patterns of duration n. Since there is one pattern (S) of duration 1, $H[1] = 1$, and since there are two patterns (SS and L) of duration 2, $H[2] = 2$. When $n > 2$, partition the collection of n-beat patterns into two sets: patterns of duration $n - 2$ followed by a long syllable and patterns of duration $n - 1$ followed by a short syllable. The number of patterns in the first set equals $H[n-2]$, since they are formed by adding L to the patterns with $n - 2$ beats, and the number of patterns in the second set equals $H[n-1]$, since they are formed by adding S to the patterns with $n-1$ beats. The partition shows that $H[n] = H[n-1] + H[n-2]$ when $n > 2$. Therefore, the list of numbers forms the Hemachandra sequence. □

Figure 2.2 gives a visual demonstration in which short and long syllables are represented by squares and dominoes, respectively. In the picture, $n = 5$, but the same argument may be made for any n greater than 2.

Exercise 2.7. The procedure for writing the patterns with duration n as a combination of patterns of durations $n-1$ and $n-2$ that is explained in the proof suggests how use the information in Figure 2.1 to list the 13 patterns with six beats. Write them.

Recursion. *Recursion* is a process in which one structure is embedded inside another similar structure, rather like nesting Russian dolls, or the Droste cocoa box. Recursion is the

[1] Fibonacci may have learned the sequence from the Indians. Fibonacci was educated in North Africa and was familiar with Eastern mathematics. His *Liber Abaci* (1202), in which the sequence appears, introduced the Indian positional number system—the system we use today—to the West. However, his description of the number sequence as counting the sizes of successive generations of rabbits is not found in India.

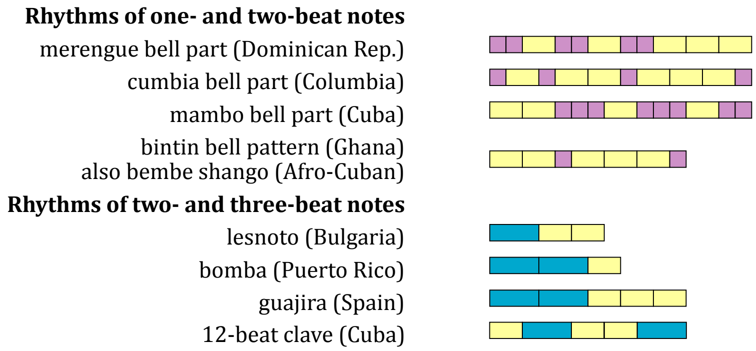


Figure 2.3: *Musical rhythms.*

lifeblood of computer programming and is crucial in mathematics as well. An algorithm is a *recursive* if you start with some information (called a *base case*) and arrive at all subsequent information by repeatedly applying the same rule, called a *recursive rule*.

In the example of the Hemachandra numbers, there are two meters of one syllable each, L and S. This is the base case. If you know all the meters that have n syllables, you can list the meters that have $n+1$ syllables by first adding an L to the beginning of the meters with n syllables, then adding an S to the beginning of the meters with n syllables. This is the recursive rule that is expressed by the formula $H[n] = H[n-1] + H[n-2]$.[2]

The Padovan sequence. The poetic meters that Pingala and Hemachandra studied have an analogue in music. Music from India, the Middle East, and the Balkans is often written in *additive meter*—that is, a rhythmic organization founded in grouping beats rather than subdividing larger units of time called measures, which is the typical structure of Western European music.

Figure 2.3 shows a few examples. The Bulgarian dance called *Daichovo horo* has a nine-beat measure, grouped 2+2+2+3. This means that the first, third, fifth, and seventh beats normally receive an accent; they are also the beats on which the dancers step. The jazz pianist and composer Dave Brubeck (1920-2012) used the same rhythm in his "Blue Rondo à la Turk" (1959). A *Gankino horo* has an eleven-beat measure, with beats grouped 2+2+3+2+2.

Many additive meters are binary patterns formed of two- and three-beat groupings. Pingala's algorithm will list all additive meters formed of n groups of beats for any n. However, musical patterns are typically classified by their number of beats. In this situation, we need something like Hemachandra's sequence for counting meters of a given duration, as explored in the following exercise:

Exercise 2.8. This problem is explored in a worksheet. The Hemachandra numbers count meters formed from one- and two-beat groups. What sequence counts meters consisting of two- and

[2] If the notation $H[n]$, $H[n-1]$, etc. is unfamiliar, it's worth taking time to understand it. Since $H[n]$ refers to the nth number in the sequence, $H[n-1]$ is the $(n-1)$th number—that is, the number preceding $H[n]$. The equation $H[n] = H[n-1]+H[n-2]$ means "the nth number is the sum of the number that is one place before it and the number that is two places before it." For example, if $n = 5$, then $H[5] = H[5-1]+H[5-2] = H[4]+H[3]$. In words, the fifth number is the sum of the fourth number and the third number. Note that $H[5-1] \neq H[5]-1$ because $H[5-1] = H[4] = 5$, while $H[5]-1 = 8-1 = 7$.

three-beat groups? Find the first few numbers in the sequence—the base case—and a recursive rule that generates the sequence. Explain why your rule is correct. This number sequence is called the *Padovan sequence* and has a rich history. Discuss. (The On-Line Encyclopedia of Integer Sequences is a wonderful research tool for this sort of problem.)

The expanding mountain of jewels

Before reading this section, I recommend trying the worksheet.

Pingala is credited with the discovery of "Pascal's" Triangle in India, which he called the *meruprastāra*, or "the expanding mountain of jewels." Mount Meru is a mythical mountain made of gold and precious stones. The nth row in this triangle counts the number of unordered combinations of n syllables: taking all n, taking $(n-1)$, taking $(n-2)$, and so on, where each syllable is considered different, rather than just long or short.

Here is how to compute the third row in the meruprastāra. There is one way of choosing all three syllables from the word "prastāra" (that is, pras+tā+ra), there are three ways of choosing two syllables (pras+tā, pras+ra, tā+ra), and there are three ways of choosing one syllable (pras, tā, ra). In order to complete the third row, note that there is one way to choose no syllables. Therefore, the third row is 1 3 3 1.

When each list counting combinations of r syllables drawn from sets of n syllables is written in a row, and the rows are stacked, the numbers form a triangular array that extends forever:

$$\begin{array}{ccccccccc} & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

Pingala recognized that each interior number is the sum of the two numbers above it. This array is known in the West as Pascal's triangle—though, of course, it wasn't yet named for Pascal, who was born in France in 1623. This triangle has long been recognized all over the world. Figure 2.4 shows three familiar images.

Exercise 2.9. Bhatt discovered the meruprastāra in a different context: he found the number of meters of n syllables having r short syllables. This is the problem that was solved in Worksheet **??**. The fact that the two problems produce the same triangle is, of course, no coincidence. Find an exact correspondence between the number of combinations of r objects drawn from a set of n different objects and the number of meters of n syllables having r short syllables.

Recursion and the meruprastāra The first row of the meruprastāra contains the numbers 1, 1. This is the base case. If you know any row in the meruprastāra, each number in the following row equals the number directly above it plus the number diagonally above and to the left (if there is no number in these positions, add zero). This is the recursive rule.

The 12th-century writer Bhaskara gives another recursive algorithm for finding the numbers in the meruprastāra in his work *Lilavati*. To find the nth row in the meruprastāra, start by

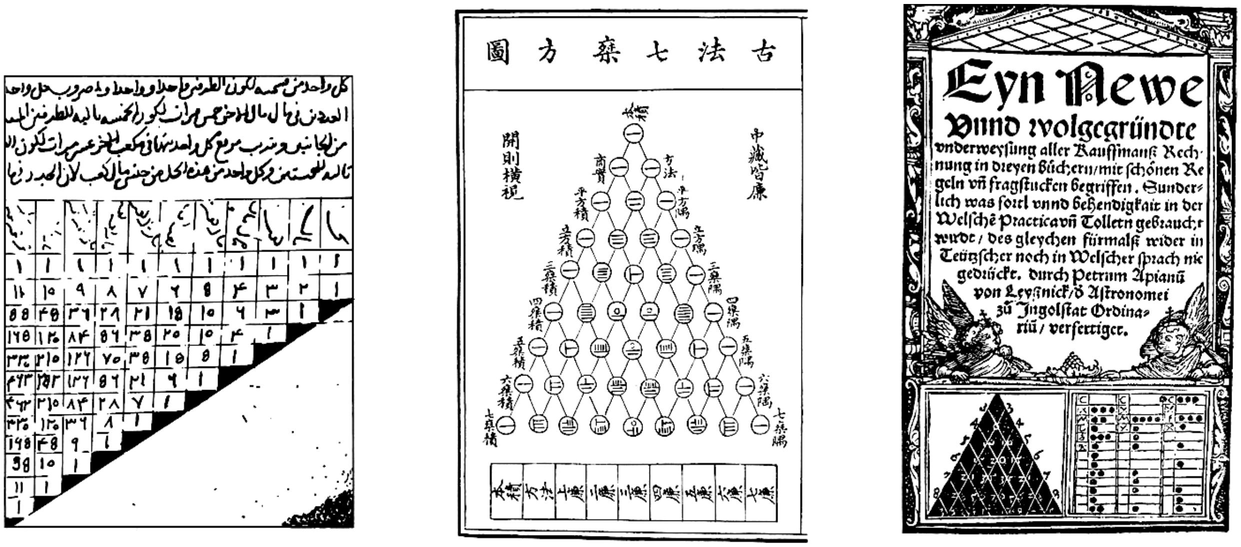


Figure 2.4: *The meruprastāra from North Africa (c.1150) to China (1303) to Germany (1527)*

writing the numbers 1, 2, . . ., n, and above them write the numbers n, $n-1$, . . ., 2, 1, like so (shown for $n = 5$):

5	4	3	2	1
1	2	3	4	5

The first number in the row is 1 (this is true for every n). Obtain the other numbers in the row by successively multiplying and dividing by the numbers you have written:

$$\frac{5}{1} = 5; \quad 5 \cdot \frac{4}{2} = 10; \quad 10 \cdot \frac{3}{3} = 10; \quad 10 \cdot \frac{2}{4} = 5; \quad 5 \cdot \frac{1}{5} = 1.$$

This tells us that the fifth row is

$$1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1$$

The numbers in row n are built up recursively, one from the next, starting from 1 (the base case).

Exercise 2.10. Find the sixth row of the meruprastāra using Bhaskara's method. Check your work using the addition algorithm, starting with the fifth row.

Patterns in music and architecture

Indian scholars had a great enthusiasm for solving mathematical problems related to poetry. However, the situation of pattern in Indian music tells a more remarkable story. There, the list of patterns has transcended its role as a "dictionary" of available patterns and has become *itself* an interesting and valuable musical structure. In music, "prastāra"—meaning systematic permutation of rhythmic elements—is commonly recognized as a principal part of the process of rhythmic variation. Indian musicians typically use prastāra towards the

<table>
<tr><td>Theme</td><td>Dhi</td><td>ne</td><td>Ta</td><td>ge</td><td>Dhi</td><td>ne</td><td>Ta</td><td>ge</td></tr>
<tr><td></td><td>ta</td><td>ke</td><td>ta</td><td>ge</td><td>Dhi</td><td>ne</td><td>Ta</td><td>ke</td></tr>
<tr><td></td><td>ta</td><td>ge</td><td>Dhi</td><td>ne</td><td>Ta</td><td>ge</td><td>te</td><td>te</td></tr>
<tr><td></td><td>ghi</td><td>de</td><td>na</td><td>ge</td><td>Tin</td><td>na</td><td>Ta</td><td>ke</td></tr>
<tr><td>Variation I.</td><td>(ta</td><td>ge</td><td>Dhi</td><td>ne</td><td>Ta</td><td>ke)</td><td>(ta</td><td>ge</td></tr>
<tr><td>6+8+6+6+6</td><td>Dhi</td><td>ne</td><td>Ta</td><td>ke</td><td>ta</td><td>ke)</td><td>(ta</td><td>ge</td></tr>
<tr><td></td><td>Dhi</td><td>ne</td><td>Ta</td><td>ke)</td><td>(ta</td><td>ge</td><td>Dhi</td><td>ne</td></tr>
<tr><td></td><td>Ta</td><td>ke)</td><td>(ta</td><td>ge</td><td>Tin</td><td>ne</td><td>Ta</td><td>ke)</td></tr>
<tr><td>Variation II.</td><td>|: (ta</td><td>ge</td><td>Dhi</td><td>ne</td><td>Ta</td><td>ke)</td><td>(ta</td><td>ge</td></tr>
<tr><td>6+6+4+6+6+4</td><td>Dhi</td><td>ne</td><td>Ta</td><td>ge)</td><td>(Dhi</td><td>ne</td><td>ta</td><td>ke) :|</td></tr>
<tr><td>Variation III.</td><td>|: (ta</td><td>ge</td><td>Dhi</td><td>ne</td><td>Ta</td><td>ge)</td><td>(Dhi</td><td>ne</td></tr>
<tr><td>6+4+6+6+4+6</td><td>Ta</td><td>ke)</td><td>(ta</td><td>ge</td><td>Dhi</td><td>ne</td><td>ta</td><td>ke) :|</td></tr>
<tr><td>Variation IV.</td><td>|: (ta</td><td>ge</td><td>Dhi</td><td>ne</td><td>Ta</td><td>ke</td><td>ta</td><td>ke</td></tr>
<tr><td>10+6+10+6</td><td>ta</td><td>ke)</td><td>(ta</td><td>ge</td><td>Dhi</td><td>ne</td><td>Ta</td><td>ke) :|</td></tr>
<tr><td>Variation V.</td><td>|: ta</td><td>ke</td><td>ta</td><td>ke)</td><td>(ta</td><td>ge</td><td>Dhi</td><td>ne</td></tr>
<tr><td>4+6+10+6+6</td><td>Ta</td><td>ke)</td><td>(ta</td><td>ge</td><td>Dhi</td><td>ne</td><td>Ta</td><td>ke :|</td></tr>
<tr><td>Variation VI.</td><td>|: (ta</td><td>ge</td><td>Dhi</td><td>ne</td><td>Ta</td><td>ke)</td><td>(ta</td><td>ge</td></tr>
<tr><td>6+10+6+10</td><td>Dhi</td><td>ne</td><td>Ta</td><td>ke</td><td>ta</td><td>ke</td><td>ta</td><td>ke) :|</td></tr>
<tr><td>Variation VII.</td><td>(ta</td><td>ge</td><td>Dhi</td><td>ne</td><td>Ta</td><td>ke)</td><td>(ta</td><td>ge</td></tr>
<tr><td>6+6+6+8+6</td><td>Dhi</td><td>ne</td><td>Ta</td><td>ke)</td><td>(ta</td><td>ge</td><td>Dhi</td><td>ne</td></tr>
<tr><td></td><td>Ta</td><td>ke)</td><td>(ta</td><td>ge</td><td>Dhi</td><td>ne</td><td>Ta</td><td>ke</td></tr>
<tr><td></td><td>ta</td><td>ke)</td><td>(ta</td><td>ge</td><td>Dhi</td><td>ne</td><td>Ta</td><td>ke)</td></tr>
</table>

Figure 2.5: *Theme and variations for tabla, as taught by Lenny Seidman. Syllables such as "Dhi" and "ne" indicate particular ways of hitting the drums' heads to produce sounds. Each syllable occupies the same amount of time. The symbols |: and :| indicate repeats and parentheses enclose phrases.*

end of a piece, since progression through all the permutations of a rhythmic pattern is a process that has a definite ending—that is, when all the possibilities have been exhausted. Lewis Rowell's description of prastāra in early Indian music is also applicable today:

> Once again we can draw an important formal conclusion from the popularity of prastāra: endings are to be signaled well in advance by the onset of some systematic musical process, a process of playful exploitation that can be followed along a course of progressively narrowed and focused expectations and that leads inexorably to a predictable conclusion. [...] But prastāra has symbolic overtones that transcend its local role as a simple tactic of closure: the device mimics the series of transformations through which all substance must eventually pass [**?**, p. 251].

Figure 2.5 demonstrates the use of permutation in a simple composition for tabla (Indian drums). Each variation on the theme is a permutation of groups of 4, 6, 8, and 10 beats:

I.	II.	III.	IV.	V.	VI.	VII.
6 8 6 6 6	6 6 4 6 6 4	6 4 6 6 4 6	10 6 10 (6	4) 6 10 6 6	6 10 6 10	6 6 6 8 6

Variations II and III state two of the three permutations of {6, 6, 4}, with 4+6+6 missing. Variations IV, V, and VI exhaust the permutations of the 10- and 6-beat phrases (note that variation V begins with the last four beats of the 10-beat phrase combining with the last 6 beats of variation VI to form the 10-beat phrase). Finally, variation VII, a mirror image of I, signals that the permutation process has come to a close.

As Rowell points out, we can also understand prastāra as manifesting a fascination with recursive generation and transformation that appears in Indian art, architecture, and religion from ancient times. The medieval Śekharī ("multi-spired") temples of western and central India gave form to the view that the cosmos was recursively generated. The eleventh-century Kandāriyā Mahādeva temple (Figure 2.6) is a celebrated example of this style; it is composed of miniature shrines (aedicules) emanating from a central shrine. Adam Hardy writes,

Figure 2.6: *Recursion in Indian architecture: the eleventh century Kandāriyā Mahādeva temple [?].*

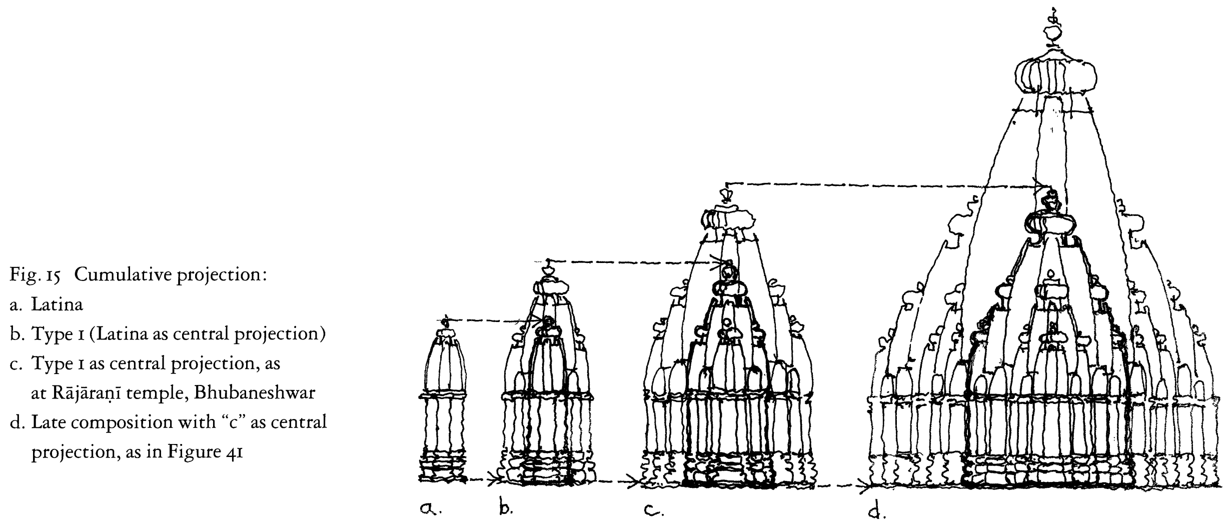


Figure 2.7: *Śekharī temple construction. ©2002 Adam Hardy*

1 1

1 1+1 1

1 2+1 1+2 1

1 3+1 3+3 1+3 1

1 4+1 6+4 4+6 1+4 1

Figure 2.8: *The meruprastāra is made of copies of itself.*

> As soon as the dynamic relationships between the aedicules are considered, the vision of a theological hierarchy can be seen as a dynamic process of manifestation: the emerging, expanding, proliferating, fragmenting, dissolving patterns are so closely analogous to the concept, perennial in India, of a world of multiplicity recurrently manifesting from unity and dissolving back into unity, that the idea can be said to be embodied in the forms [**?**, p. 91-2].

It is no wonder that ancient and medieval Indian mathematicians developed an outstanding facility with recursion.

Solutions to exercises

Solution 2.1. I've written the list in two columns to save space.

LLLLL
SLLLL
LSLLL
SSLLL
LLSLL
SLSLL
LSSLL
SSSLL
LLLSL
SLLSL
LSLSL
SSLSL
LLSSL
SLSSL
LSSSL
SSSSL
LLLLS
SLLLS
LSLLS
SSLLS
LLSLS
SLSLS
LSSLS
SSSLS
LLLSS
SLLSS
LSLSS
SSLSS
LLSSS
SLSSS
LSSSS
SSSSS

Solution 2.2. Since LLSLSL has seven syllables, the list has $2^7 = 128$ meters.

Solution 2.3. Pingala's algorithm means that each list is twice the size of the previous one (formally, if there are k $(n-1)$-syllable meters, there are $(2 \times k)$ n-syllable meters. Since there are 2 one-syllable patterns, the numbers of meters of each length follows the pattern 2, 4, 8, 16, and so on.

Solution 2.4. Since patterns of H and T are binary, there are $2^3 = 8$ patterns, and they are HHH THH HTH TTH HHT THT HTT TTT

Each of these patterns is equally likely, and three of them have one T and two H's, so the likelihood of getting one T is 3/8 or 37.5%. In general, use the results from the study of meters, substituting H for L and T for S.

Solution 2.5. Start with 1, then use the list of three-syllable patterns, replacing 0 with L, 1 with S, and writing the patterns backwards. There are $2^4 = 16$ five-digit binary numbers. You write them by following this same procedure with the list of four-syllable meters. Proving that this answer is correct involves understanding how place-value number systems work in bases other than 10.

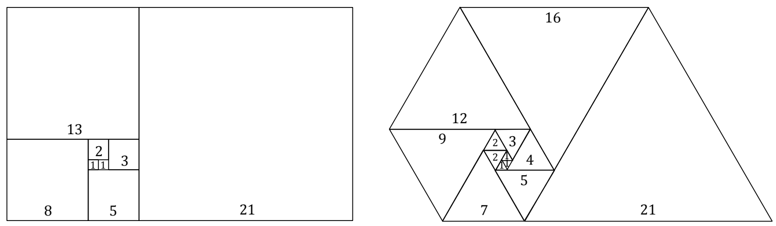
Solution 2.6. If your conjecture was something like "there are 13 meters with 6 beats, and you get any number in the sequence by adding the two previous," you would be correct.

Solution 2.7. The procedure is to add a L to all the 4-beat patterns, then add an S to all the 5-beat patterns, which are listed in figure 2.1. The answer is: LLL SSLL SLSL LSSL SSSSL SLLS LSLS SSSLS LLSS LLSLS SLSSS LSSSS SSSSSS

Solution 2.8. Here are the first ten entries of the Padovan sequence:

duration	1	2	3	4	5	6	7	8	9	10
num. patterns	0	1	1	1	2	2	3	4	5	7

If $P[n]$ is the number of n-beat patterns, then a recursive rule is $P[n] = P[n-2] + P[n-3]$ when $n > 3$. The proof of this statement is similar to the proof of Theorem 2.2 (the Hemachandra numbers). In this case, partition the patterns of duration n into patterns of duration $n-2$ followed by a two-beat note and patterns of duration $n-3$ followed by a three-beat note. Incidentally, a student noticed that $P[n] = P[n-1] + P[n-5]$ (when $n > 5$) and conjectured that all Padovan numbers also follow this rule. Is this correct? Hint: use the first rule to rewrite $P[n-1]$. The Padovan sequence has some beautiful properties—for example, it is related to a spiral of equilateral triangles in the way the Hemachandra-Fibonacci sequence is related to a spiral of squares (see below), and it is closely connected to the Perrin sequence, another extremely cool sequence. See Ian Stewart's "Tales of a Neglected Number" (in *Math Hysteria*) for more examples.



Solution 2.9. Any way of choosing objects from a collection of n different objects can be represented by a binary pattern of length n in this way: assign numbers from 1 to n to the objects, and write I (in) if an object is selected and O (out) if it is not. For example, suppose you choose $\{2, 5, 7\}$ from the collection $\{1, 2, \ldots, 8\}$. That choice corresponds to the binary pattern OIOOIOIO. Each choice of r objects corresponds to a pattern of n letters with r I's. Substituting S for I and L for O produces a n-syllable meter with r short syllables.

Solution 2.10.

$$\begin{array}{cccccc} 6 & 5 & 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{array}$$

The first number in the row is 1 (this is true for every n). Obtain the other numbers in the row by succesively multiplying and dividing by the numbers you have written:
$\frac{6}{1} = 6; 6 \cdot \frac{5}{2} = 15; 15 \cdot \frac{4}{3} = 20; 20 \cdot \frac{3}{4} = 15; 15 \cdot \frac{2}{5} = 6; 6 \cdot \frac{1}{6} = 1$. This tells us that the sixth row is
$1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1.$

Chapter 3

Rhythm

The beat is the foundation of music. Even before birth, we hear the regular pulse of our mothers' heartbeats. We experience the beat as simultaneously linear (progressing through time) and cyclic (repeating, as on a clock). Musical events—sounds and silences—occur within a repeating framework called a meter. A rhythm is a pattern of note onsets that are actually present in a piece. In traditional and popular music, repeated rhythmic patterns, or time-lines, overlay the meter.

Measuring time

Watch the video "Tāla in Carnatic classical music." In this video, the performer's repeated hand gestures indicate the tāla, or repeating rhythmic organization of time in the piece. These same gestures accompany any piece using this tāla. Compare also the hand gestures used in Western classical conducting (Figure 3.1).

The *tactus* is the basic pulse of a piece of music—it's where you naturally tap your foot. In music with a regular rhythm, each pulse, or *beat*, has the same length. When tapping along with the tactus, we normally expect some musical "event" to happen at every beat we tap. Normally this event is something we can hear (a drum hit, or the beginning of a note) but sometimes there is a silence—a *rest*. The *tempo*, or speed, of the tactus is measured in beats per minute. Tempos of about 120 beats per minute are typical for pop songs, while a comfortable walking tempo is around 100 beats per minute. A *metronome* is a simple mechanical or digital instrument that can play beats at a chosen tempo.

Beats in the tactus are grouped into units called *measures*, or *bars*, just as seconds are grouped into minutes. Beats may also be subdivided into smaller units of time. In Western classical and popular music, divisions into two, three, or four equal parts are common. This diagram shows a four-beat measure with beats subdivided into two or four parts. The "&" should be read as "and." Practice clapping at a constant rate on the numbered beats and repeat each line of the drill several times along with your claps. Then practice three-beat measures, using the patterns 123, 1&2&3&, and 1e&a2e&a3e&a.

In some pieces, there are intermediate levels of grouping between the measure and the tactus and, in addition, higher-level groupings of measures. In Western music notation, the *time signature* indicates the internal structure of the measure. For example, the time

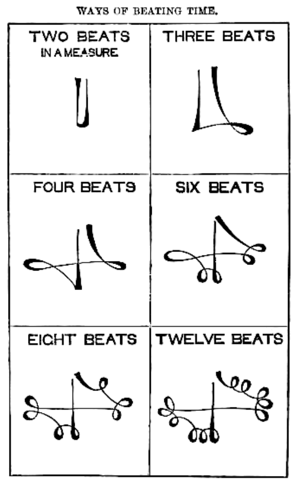


Figure 3.1: *Curwen's conducting diagrams.*

1				2				3				4			
1		&		2		&		3		&		4		&	
1	e	&	a	2	e	&	a	3	e	&	a	4	e	&	a

Figure 3.2: *Duple and quadruple subdivisions of a four-beat measure.*

signature 12/8 is normally played as four beats per measure, each subdivided into three parts. H.W. Day's "Tree of Time" (Figure 3.4) is a fanciful representation of time signatures.

Exercise 3.1. Practice clapping to three songs written: "Respect," by Otis Redding (1965; made famous by Aretha Franklin), the Beatles' "Norwegian Wood" (1965), and Sister Rosetta Tharpe's "That's All" (1939). First, find the tactus. It should be where you naturally step when you're dancing. Are beats in the tactus grouped any particular way, such as into groups of two, three, or four? Can those groups be grouped into larger groups? Next, try dividing each beat in the tactus into two subdivisions. If that doesn't seem to work, try three subdivisions. Describe the rhythmic structure of the piece.

Rhythm and groove

Groove is a difficult term to define. Most musicians agree that groove is essential to most styles of popular music. It normally refers to characteristic repeating rhythm patterns and accents that identify different styles of dance music, such as ska and reggae. Listen to the audio, starting around 3:00, in which James Brown, "The Godfather of Soul," discusses grooves in his music in this 2005 interview with Terry Gross of NPR. He compares two versions of the "I Got You" groove and describes how his groove changed with his 1965 song "Papa's Got A Brand New Bag."

The first count in every measure is called the *downbeat* and the last count is called the *upbeat*, because it comes right before the downbeat of the next measure. The most basic

1			2			3			4		
1	e	a	2	e	a	3	e	a	4	e	a

Figure 3.3: *Triple subdivisions of a four-beat measure.*

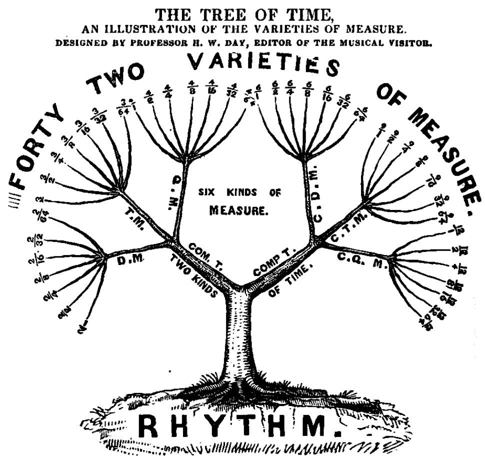


Figure 3.4: *A fanciful representation of meter in Western classical music.*

grooves in Western music use four-count measures. Counts one and three are called the *on-beats* and counts two and four are the *off-beats*. "Backbeat" grooves, popular in blues and R&B, emphasize the off-beats, while downbeat-oriented grooves have a primary accent on the downbeat and a secondary accent on count three. In the NPR interview, James Brown said that "Papa's Got a Brand New Bag" was his first song to give a strong emphasis to the downbeat.

Unlike the basic grouping of pulses in the tactus into measures, a groove can be highly complex, with several musicians playing interlocking rhythms, each with different accent patterns. A hip-hop beat is an example of what I'm calling a groove. We can study both the pattern that each instrument plays and the combination of all these patterns. In addition, musicians sometimes intentionally play a little ahead or behind the tactus—this kind of variation is called *microtiming* or *swing*. Although microtiming is an important feature of dance music grooves, it is difficult to write down in music notation. Musicians normally learn it by ear. Just to keep things simple, we won't be studying microtiming in this class, but it is an important part of groove in a lot of music.

Exercise 3.2. Practice both on-beat and off-beat clapping with different kinds of dance music. Find a piece that clearly emphasizes the on-beats and another piece that clearly emphasizes the off-beats. Do you prefer to clap to "Papa's Got a Brand New Bag" on the on-beats, or on the off-beats?

Notation

Drum tablature, or drum tab, is a common way to notate rhythm. Sequences of drum hits can be written with `x`'s, indicating hits, and `.`'s (periods), or rests, indicating that the drum is not sounded. Each symbol occupies the same amount of time, a pulse in either the tactus or some regular subdivision of the tactus. For example, the notation `x..x..x.` means "hit rest rest hit rest rest hit rest." This is a common pattern in music with four counts to a measure, subdivided once. Practice clapping the pattern while you speak the names of the counts:

```
Count: 1 & 2 & 3 & 4 &
Clap:  x . . x . . x .
```

Exercise 3.3. Write the pattern for a four-count measure, with hits (a) on the downbeat only (b) on the upbeat only (c) on the on-beats, and (d) on the off-beats. Each `x` or `.` occupies one count.

Exercise 3.4. How many patterns of hits and rests are possible in an eight-beat measure? Of those, how many have three hits? (Don't try to do this from scratch—use the results of the last chapter!)

Steve Reich's "Clapping Music." We can use drum tab to visualize Steve Reich's "Clapping Music." Player A claps the same rhythm over and over until the piece ends.

```
Player A:     x x x . x x . x . x x .
```

Player B progresses through a number of patterns, playing each one eight times. The first pattern is the same as Player A's, then the pattern is repeatedly shifted by one beat, so that, for example, `xx.xx.x.xx.x` is the next pattern.

When two drum patterns are played simultaneously, a third rhythm is heard, called the *resultant rhythm*. It is the rhythm that has rests whenever both of the original drum patterns had a rest and drum hits otherwise.

Drum tab is convenient for finding resultant rhythms. Any beat that has an `x` in either A or B (or both) has an `x` in the resultant rhythm. Here are patterns A and B in the previous example, plus the resultant rhythm. The seventh beat is the only rest in the resultant rhythm.

```
Player A:         x x x . x x . x . x x .
Player B:         x . x x . x . x x . x x

resultant:        x x x x x x . x x x x x
```

Led Zeppelin's "Kashmir" Here is the Kashmir pattern. The drums are hit on the indicated counts.

```
Count:   1 & 2 & 3 & 4 & |1 & 2 & 3 & 4 & |1 & 2 & 3 & 4 & |1 &  etc.
Guitar:  xxx...xxx...xxx.|..xxx...xxx...xx|x...xxx...xxx...|xxx. etc.
```

Exercise 3.5. Explain why the overall pattern, including both drum and guitar, repeats after three measures.

Circular notation.

Because grooves repeat, it sometimes makes sense to visualize patterns on a circle, just as we visualize hours on a clock. The downbeat goes at the top and the other beats proceed

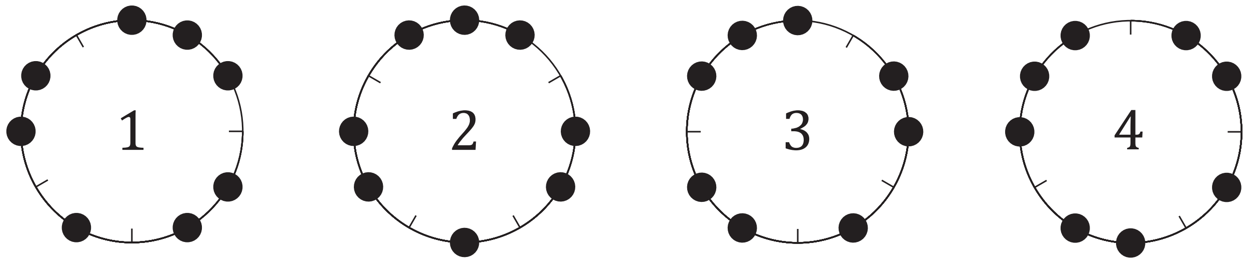


Figure 3.5: *The first four patterns in "Clapping Music" that are clapped by player B.*

clockwise. Dots show when the drum is hit. Figure 3.5 shows some of the patterns from "Clapping Music."

"Scientific" time.

In music, time is of the essence. How should we measure it? Rather than using minutes or seconds, we'll use a unit of time that is relevant to the piece itself. Since the tactus is often subdivided, using the tactus as a "ruler" to measure time would mean using a lot of fractions. To avoid this, let's measure time using the smallest level of subdivision, called the *tatum*, in honor of the great African-American jazz pianist Art Tatum. Think of a tatum as the musical version of an atom. In music notation, the tatum might be a pulse of eighth or sixteenth notes. Each pulse in the tactus lasts a whole number of tatums.

The fact that a tatum, like an hour or a minute, is a measure of time—not a specific point in time—means that we can measure musical time starting at zero, just like a ruler starts at zero, so that "time 0," or t0, refers to the beginning of the first measure and "time a," or ta, refers to the time a tatums later. The first beat is the beat whose onset is at t0.[1]

Any pattern of hits corresponds to a pattern of time intervals (durations) that measure the time from the onset of one hit to the next. For example, the pattern `x..x..x.x..x..x.` is described as "3+3+2+3+3+2" and shown in Figure 3.6. The onsets in the pattern start at t0, t3, t6, t8, t11, and t14.

The multiple levels of counting in Figure 3.6 give some idea of the complexity of musical time. The tatum is, here, twice as fast as the tactus. If the tatum and the tactus are the same length, I'll just refer to "the beat" rather than distinguishing between levels of subdivision.

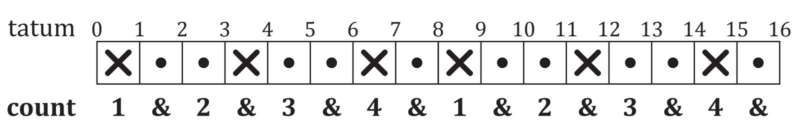


Figure 3.6: *Two repeats of the pattern* `x..x..x.`

[1] An analogy is that the first hour after midnight is the hour between 0:00 and 1:00, in a 24-hour clock.

The beat class circle and modulus

When we translate between the beat and the measure, certain points in time are associated with one another. Just as a clock represents the organization of time within a day, the *beat class circle* represents time in a musical measure. Figure 3.7 shows a repeated rhythm pattern, called a *timeline*, on a beat class circle with eight beats. The blobs indicate which time points are attacks.

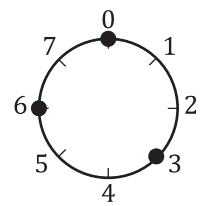


Figure 3.7: *The timeline* `x..x..x.` *represented on the beat class circle.*

Definition 3.1 (Beat class). *Suppose there are n beats per measure and a is one of the numbers $\{0, 1, 2, \ldots, (n-1)\}$. Then beat class a (bc$a$) equals the set of all time points that are a beats after the start of some measure.*

A measure with n tatums has n beat classes,

$$\text{bc}0, \text{bc}1, \text{bc}2, \ldots, \text{bc}(n-1)$$

Beat class bc0 consists of the onsets of all the downbeats (equivalently, you can think of bc0 as the vertical lines indicating the beginnings of measures).

Exercise 3.6. Write the rhythms in Figure 2.3 (see page 2–6) in drum tab and circular notation and write the set of beat classes in which their onsets fall.

Suppose you start at the beginning of a measure and there are 12 beats per measure. What is the beat class at time t54? Since moving backward 12 beats has no effect on the position in the measure, we start by subtracting as many 12s as we can, then looking at what's left. Since $54 - 12 - 12 - 12 - 12 = 6$, beat 54 is in beat class 6. It occurs 6 beats after the beginning of a measure. We can do this more efficiently using remainders: 6 is the remainder when 54 is divided by 12.

In general, suppose there are n beats in a measure. Time point ta falls in beat class R, where R is the remainder when a is divided by n. Mathematicians call this remainder the *modulus*.

Definition 3.2 (Modulus). *If a is an integer and n is a positive integer, then the modulus of a relative to n is the remainder when a is divided by n. If R is the remainder, we write*

$$a \text{ mod } n = R$$

Theorem 3.1 (Beat class equals modulus). *If there are n beats per measure, time point ta belongs to beat class a mod n.*

Two time points belong to the same beat class if and only if they are in the same position in the measure, which is the same thing as being separated by a whole number of measures. Mathematicians call this relationship *modular congruence*:

Definition 3.3 (Modular congruence)**.** *Two numbers a and b are congruent modulo a positive integer n if*

$$a \bmod n = b \bmod n$$

If a and b are congruent, we write $a \equiv b \pmod{n}$.

The triple equals sign $\equiv$ is read as "is congruent to." For example $23 \equiv 51 \pmod{7}$ because the remainder when either 23 or 51 is divided by 7 equals 2. That is, $23 \bmod 7 = 51 \bmod 7$. The following theorem shows an alternate formula used to determine modular congruence.

Theorem 3.2. *If a and b are integers and n is a positive integer, then $a \equiv b \pmod{n}$ if and only if $(a - b)$ is divisible by n; that is, $(a - b)/n$ is an integer.*

For example, $23 \equiv 51 \pmod{7}$ because $(51 - 23)/7 = 28/7 = 4$, which is an integer.

Theorem 3.3 (The beat class theorem)**.** *Suppose there are n beats in a measure. Time points ta and tb lie in the same beat class if and only if*

$$a \equiv b \pmod{n}$$

which occurs if and only if $(a - b)/n$ is an integer.

For example, suppose there are seven beats to a measure. Then

- Beat class 0 corresponds to the onset of downbeats, which begin at t0, t7, t14, t21 ....
- Time t100 falls on beat class 2 because $100 \div 7 = 14r.2$.
- Time t28 falls on beat class 0 because 28 is divisible by 7 ($28 \div 7 = 4r.0$).
- Times t100 and t849 belong to the same beat class because they are separated by $849 - 100 = 749$ beats, which equals exactly $749/7 = 107$ measures. Equivalently, use modular congruence and the beat class theorem: $100 \equiv 849 \pmod{7}$ because $(100 - 849)/7$ is an integer. Another way to solve the problem is to use modulus: t100 is in bc 2 (see above) and, since $849 \div 7 = 121r.2$, t849 is also in bc 2.

Exercise 3.7. Evaluate (a) $144 \bmod 13$ and (b) $169 \bmod 13$.
Determine whether the equations are true:
(c) $25 \equiv 61 \pmod{6}$
(d) $-4 \equiv 4 \pmod{3}$.

Exercise 3.8. Suppose there are six beats to the measure. Which time points are downbeats? To which beat class does time t50 belong? Time t90? Are time points 36 and 63 in the same beat class?

Exercise 3.9. Suppose there are four beats per measure and you clap on every third beat, starting at the beginning of a measure. Find the sequence of beat classes on which your claps fall. Practice counting in four and clapping every third beat.

Solution 3.3. (a) `x...` (b) `...x` (c) `x.x.` (d) `.x.x`

Solution 3.4. Hits and rests are binary patterns, and the number of binary patterns of length n is 2^n. Therefore, there are $2^8 = 256$ patterns. The eighth row of Pascal's triangle tells us that there are 56 eight-beat patterns that have three hits and five rests.

Solution 3.5. The drum pattern repeats every measure (eight beats) and the guitar pattern repeats every three beats. The entire pattern repeats after lcm$(8, 3) = 24$ beats, which equal three measures.

Solution 3.6. In general, a duration of one is written "`x`"; two is written "`x.`", and three is written "`x..`". For example, the guajira is `x..x..x.x.x.` (this is the rhythm of "America" in West Side Story). The onsets fall on beat classes 0, 3, 6, 8, and 10.

Solution 3.7. (a) $144 \bmod 13 = 1$ because $144 \div 13 = 11r.1$ and (b) $169 \bmod 13 = 0$ because $169 \div 13 = 13$ (remainder is 0). Determine whether the equations are true:
(a) $25 \equiv 61 \pmod 6$ because $(25 - 61)/6 = 6$, which is an integer.
(b) $-4 \not\equiv 4 \pmod 3$ because $(-4 - 4)/3 = -8/3$ is not an integer.

Solution 3.8. The downbeats occur at time points 0, 6, 12, 18, etc. Time 50 is in beat class 2 because $50 \div 6 = 8r.2$. Time 90 is in beat class 0 because $90 \div 6 = 15r.0$.

Solution 3.9. You clap on beat classes
$0 \bmod 4 = 0$
$3 \bmod 4 = 3$
$(3 + 3) \bmod 4 = 2$
$(2 + 3) \bmod 4 = 1$
$(1 + 3) \bmod 4 = 0$, etc.
So you clap on beats 0, 3, 2, 1, 0, 3, 2, 1, ...

Polyrhythm and maximally even rhythms

I've presented the m-against-n rhythm as one person counting in cycles of m beats and the other person counting in cycles of n beats. We saw that a m-against-n pattern repeats after lcm(m, n) beats. What more normally happens is that a musician plays a repeating pattern whose cycle length is not a divisor or multiple of the number of beats in a measure, while, at the same time, keeping track of the underlying meter. An example is the Kashmir pattern, which has three beats, where there are eight beats per measure. Eight repeats equal twenty-four beats and fill a whole number of measures. This is expressed mathematically as $(3 \cdot 8) \bmod 8 = 0$. Note that $24 = $ lcm$(3, 8)$. This 3-against-8 rhythm is an example of a *polyrhythm*.

Definition 3.4. *Suppose n and m are positive integers. A m-against-n polyrhythm is either the result of repeating a pattern of length n beats simultaneously with a pattern (or measure) of length m beats.*

In general, assuming that n and p are positive, repeating a pattern whose duration is p beats r times takes rp beats. This equals a whole number of n-beat patterns or measures if

$$rp \bmod n = 0$$