Chapter 3

Pitched sounds

Sound is vibration that travels through air or water. When we refer to "sound," we normally mean "audible sound"—that is, vibration that our ears can detect. Pitched sounds, or tones, are present in most music. Loosely, a sound is pitched if you can hum along with it. The human voice and most musical instruments, with the exception of some drums, can produce pitched sounds. Physics and mathematics can explain not only the characteristics of pitched sounds and how musical instruments produce them but also which combinations of tones will seem most harmonious.

3.1 Periodicity

What is special about pitched sounds? Listen to the sound on the web site of a cellist playing the open strings of the instrument. We can open the cello recording in Audacity and zoom in on the waveform. What we see is a graph, with time in seconds on the horizontal axis and displacement on the vertical axis. The computer reads the displacement measurements and tells the speakers how to vibrate so you can hear the sound. These vibrations travel through the air and are interpreted by our ears.

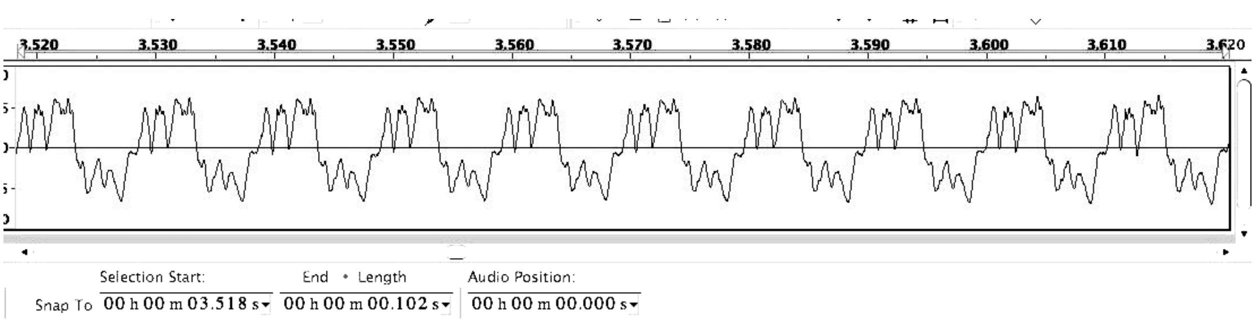


Figure 3.1: *A 0.1 second portion of the waveform of a cello's sound.*

Figure 3.1 shows a 0.1 second portion of the waveform of the cello sound. This waveform appears to repeat the pattern in Figure 3.2 over and over.

Definition 3.1. *A pattern occurring in time is periodic if it repeats the same pattern at*

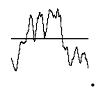


Figure 3.2: *One cycle of the cello's waveform.*

regular time intervals. Its period is the length, in seconds per cycle, of the smallest portion of the overall pattern that can be repeated to form the overall pattern. A cycle is a portion of the overall pattern whose length equals one period.

Example: a sawtooth wave. Figure 3.3 depicts a *sawtooth wave*, which you can hear on the web site. It sounds artificial but is still is a pitched sound. This computer-created sound is perfectly periodic, while the cello's waveform is only approximately periodic.

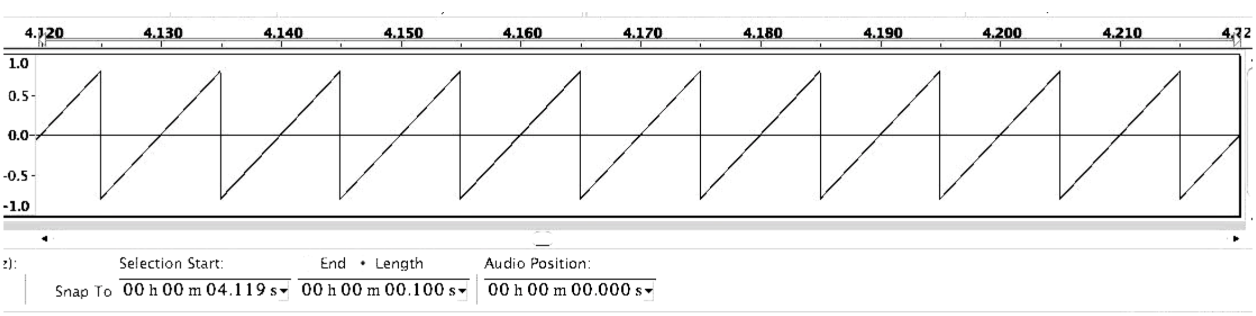


Figure 3.3: *A 0.1 second portion of a sawtooth wave.*

The length of the clip shown is exactly 0.1 second. However, a shorter pattern—a cycle—can be repeated to make the entire clip. There is no unique cycle, as any portion of the waveform that lasts one period is a cycle. Three examples of cycles are shown in Figure 3.4. Each cycle lasts 0.01 second, so the period of the vibration is 0.01 seconds per cycle.

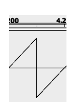
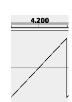
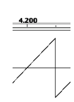
this  or this  or this 

Figure 3.4: *Three cycles of the sawtooth wave (Figure 3.3).*

Example: white noise. Of course, not every waveform is periodic. A sound produced by random vibrations is called *white noise*, as shown in Figure 3.5. It sounds like hissing or static and is not pitched. You can listen to this clip on the web site.

Exercise 3.1. Estimate the period of the vibrations of the cello sound. Use correct units in your answer.

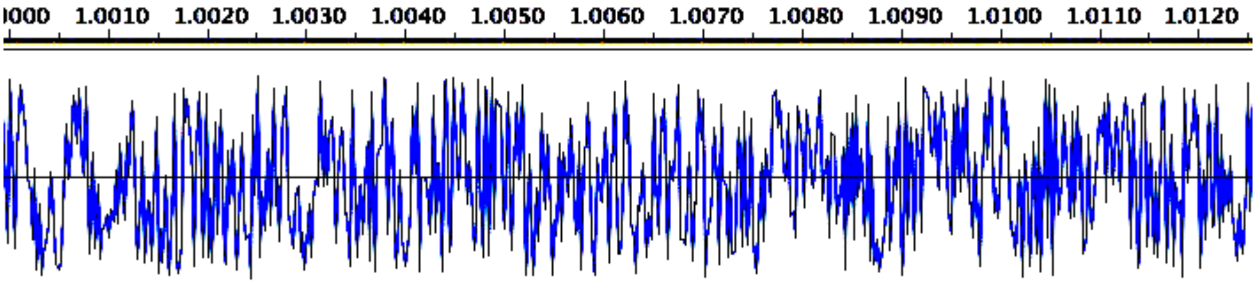


Figure 3.5: *The waveform of a white noise.*

3.2 Frequency

Pitched sound is sound produced by rapid, regular vibration. Frequency measures the speed of vibration.

Definition 3.2. *The frequency of a periodic vibration is its number of cycles per second. The units of frequency are Hz, or Hertz.*

Humans can hear pitched sounds that range from about 20 Hz to 20,000 Hz. Other animals can hear sounds with lower or higher frequencies.

Period-frequency conversion. The frequency and period of any periodic waveform are reciprocals.[1] That is,

$$\text{frequency} = \frac{1}{\text{period}} \quad \text{and} \quad \text{period} = \frac{1}{\text{frequency}}.$$

where the frequency is measured in Hertz (cycles per second) and the period is measured in seconds per cycle. Note that the units of measurement are also reciprocals.

Example: using frequency to generate a tone. Let's make a tone like the sawtooth wave that we heard. In order to use Audacity's sound generator, I need to know the frequency of the vibration. I calculated that the period of the sawtooth wave is 0.01 second. That is, the vibration is

$$0.01 \text{ seconds per cycle,} \quad \text{or} \quad 0.01 \tfrac{\text{seconds}}{\text{cycle}}.$$

However, frequency is measured in cycles per second, or $\frac{\text{cycles}}{\text{second}}$. Take the reciprocal to find the frequency.

$$0.01 \tfrac{\text{seconds}}{\text{cycle}} \text{ corresponds to } \tfrac{1}{0.01} \tfrac{\text{cycles}}{\text{second}} = 100 \text{ Hz}.$$

[1]If $n \neq 0$, the numbers n and $1/n$ are *reciprocals*. For example, $1/3$ and 3 are reciprocals.

Now go to Generate → Tone in Audacity and make a 100 Hz sawtooth wave.

Pitched sound. I informally described a "musical tone" as "a sound you can hum along with." Instruments such as a violin, piano, guitar, or trumpet produce such sounds, as does a person singing or whistling. We looked at the waveforms of recordings of musical instruments and saw that they were created by periodic, or repeating, vibrations. You might wonder if *any* periodic vibration can be described as "musical." The answer is no, because vibrations that are either very slow or very fast cannot be heard.

Definition 3.3. *A pitched sound is an audible sound produced by a periodic vibration.*

A few observations…

- If two pitched sounds have the same frequency, they are said to have the same *pitch*.
- If one sound has a higher frequency than another, it is said to have *higher pitch*, or to be a *higher* sound.
- If one sound has a lower frequency than another, it is said to have *lower pitch*, or to be a *lower* sound.
- Because frequency and period are reciprocals, high pitches correspond to short periods, while low pitches correspond to long periods.

Exercise 3.2. Use your estimate of the cello wave's period to estimate the frequency of the cello sound. Make a sound with that frequency in Audacity. Which of the four cello pitches is it most like?

Exercise 3.3. What is the period of the lowest audible sound? Of the highest audible sound?

Exercise 3.4. Suppose a violin plays a sound whose period is 0.002 seconds and a clarinet plays a sound whose period is 0.0025 seconds. Compute the frequency of each sound. Which sound is lower?

Exercise 3.5. Generate or record a few seconds of pitched sound. In Audacity, select half of the waveform and use the Effects → Change Pitch option to double its frequency. Listen to the entire clip. Now, generate a white noise and do the same thing. How does white noise behave differently from pitched sound?

3.3 Amplitude and envelopes

So far, we have seen that pitched sounds are periodic, at least over small intervals of time. Another quantity that affects what you hear is volume. Louder sounds generally result from more powerful vibrations, and therefore the waveform is wider. A waveform's *amplitude* is half its width (that is, half the difference between its maximum and minimum points), which normally is equal to its height above the horizontal axis. Increasing a waveform's amplitude increases how loud it sounds. Here is a five-second clip from ZZ Top's song "Sharp-Dressed Man." The wave gets wider on the loud drum beats, which occur about twice a second.

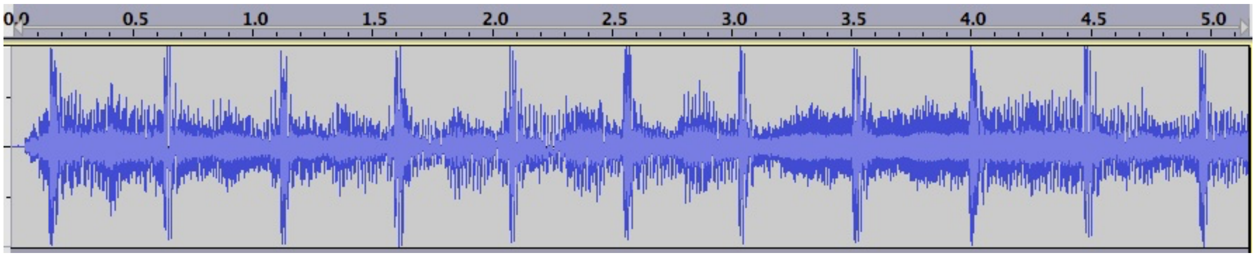


Figure 3.6: *Five seconds of ZZ Top's "Sharp-Dressed Man."*

An *envelope* is a representation of how a sound's amplitude changes over time. Its graph has roughly horizontal symmetry, with the top curve tracing the maximum points in the waveform and the bottom curve tracing the minimum points. If a sound fades out, its envelope will pinch together like a ">" symbol. The envelope often gives information about the rhythm, since hitting a drum produces a sudden change in volume. The outer curves in Figure 3.7 trace the envelope of the ZZ Top song "Sharp-Dressed Man."

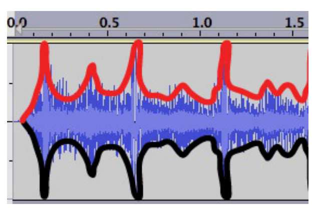


Figure 3.7: *Envelope of the beginning of "Sharp-Dressed Man."*

Is amplitude the same thing as loudness? The short answer is no. *Volume*, or perceived loudness, is a complex psychological phenomenon determined by the frequency and quality of a sound, as well as its amplitude. There is no widely-used method to measure volume. However, if two sounds have the same frequency and are played on the same instrument, the sound with a greater amplitude will be perceived as louder.

Exercise 3.6. The Envelope Tool on Audacity lets you modulate the volume of a sound wave (look for the button on the top of the window). Clicking and dragging shapes the envelope of the sound wave. I have produced a waveform with Audacity by generating five seconds of a 440 Hz tone and using the Envelope tool to modify it (Figure 3.8). Sketch the envelope of this waveform. Describe what you would hear when the sound is played.

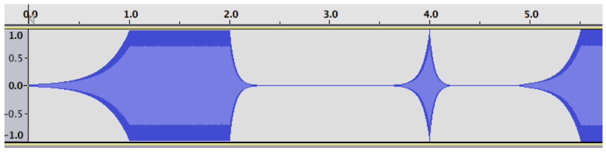


Figure 3.8: *Waveform produced using the Audacity Envelope tool.*

3.4 Sinusoids

When you strike a tuning fork, its tines vibrate at 440 Hz. If you attach a fine pen to the fork and move it while it is vibrating, you can sketch a smooth wave, as in Figure 3.9. In class, we used Audacity to record the sound of a tuning fork. Its waveform, shown in Figure 3.10, is a *sinusoid*, which is a curve that has the same basic shape as the graph of $y = \sin(t)$ (Figure 3.11). Sounds with sinusoidal waveforms are often described as "pure" tones, like the sound of a tuning fork. Sinusoids also represent the oscillation (back and forth movement) of a spring or pendulum.

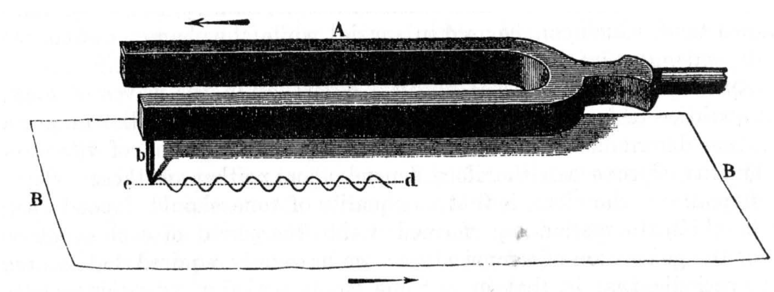


Figure 3.9: *A tuning fork's vibration produces a sinusoidal pattern.*

Not all sinusoids have the same frequency or amplitude. A sine wave with amplitude equal to A and frequency equal to f Hz has the equation

$$y = A\sin(2\pi ft)$$

where the sine is calculated in radians and t is time in seconds. The general formula for a sinusoid is $y = A\sin(2\pi ft + \varphi)$, where φ is the wave's *phase*, which is a number between 0 and 2π that indicates where in the wave's cycle the time $t = 0$ falls. Changing the phase moves the wave horizontally—that is, back and forth in time. The phase has no effect on the sound of a single waveform, but becomes important when waveforms are combined.

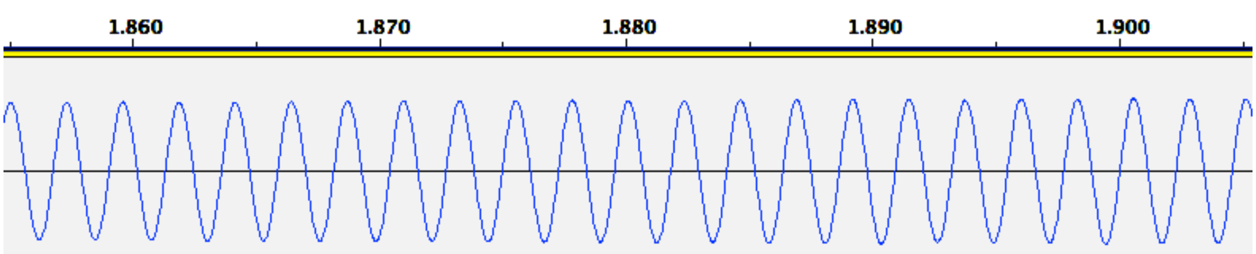


Figure 3.10: *Waveform recorded from a tuning fork.*

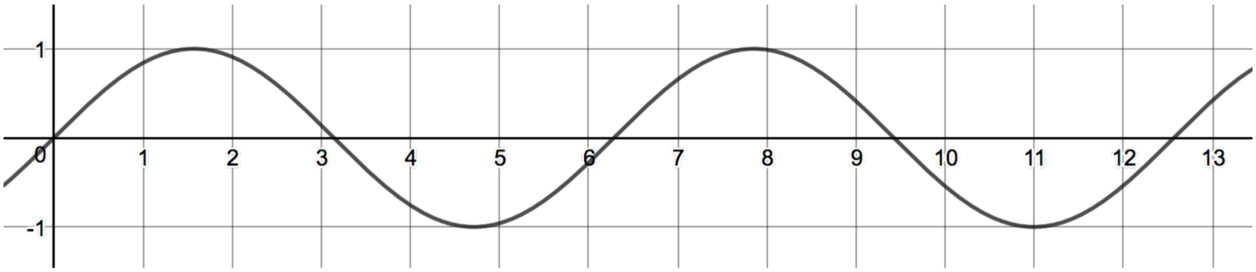


Figure 3.11: *Graph of $y = \sin(t)$. It crosses the horizontal axis at $t = 0, \pi, 2\pi, 3\pi$, etc.*

Example. Figure 3.12 shows the graph of a sinusoid with an amplitude of 0.8 and a period of 0.01 seconds. Suppose I want to know its formula, in the form $y = A\sin(2\pi ft)$. The value of $A = 0.8$ is given. I then calculate the frequency $f = 1/P$. Since $P = 0.01$ seconds, $f = 1/0.01 = 100$ Hz. This wave is the graph of $y = 0.8\sin(2\pi \cdot 100 \cdot t) = 0.8\sin(200\pi t)$. Although it is true that π can be approximated numerically, it is preferred to write 200π rather than something like 628.3185, which approximates 200π to four decimal places, while 200π is an exact answer.

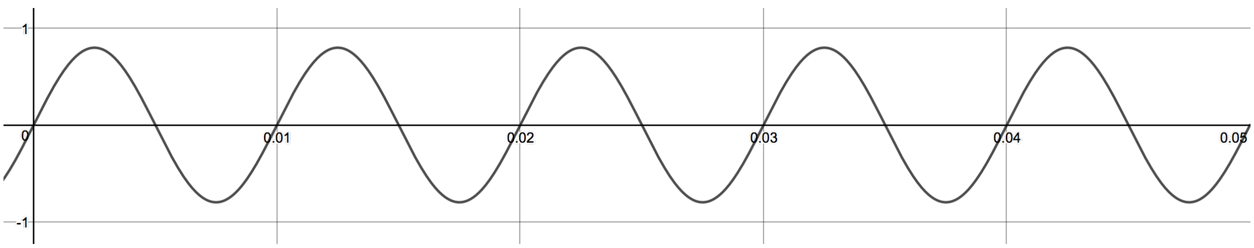
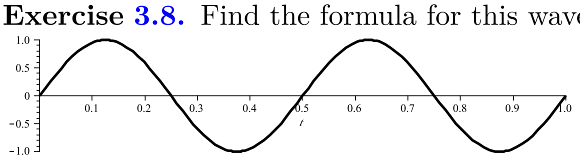


Figure 3.12: *A sinusoid with amplitude = 0.8 and period = 0.01 seconds.*

Exercise 3.7. Find the amplitude, frequency, and period of the waveform represented by $0.4\sin(100\pi t)$. Use correct units in your answer.

Exercise 3.8. Find the formula for this wave. The numbers on the horizontal axis are seconds.



Exercise 3.9. Sketch a sine wave with amplitude = 3 and period = 4 seconds. Be sure to label both the horizontal and vertical axes of your graph. Find the formula for this wave and use a graphing program to check your answer.

Check out Worksheet **??** for more practice problems.

3.5 The Superposition Principle

The *Superposition Principle* explains how multiple sounds combine. It is the result of a basic principle of physics stating that the impact of two forces at a point is the sum of those forces.

Definition 3.4. *The Superposition Principle states that the waveform of two sounds that reach the same place at the same time is the sum of the waveforms of each sound on its own.*

"Mixing" and addition of waveforms. When adding waveforms, you add their y-coordinates. Recording engineers call this process *mixing*. In Audacity, if you select two waveforms, then choose Tracks → Mix and Render To New Track, the sum of the waveforms will appear in a new track. Figure 3.13 shows the result of mixing a 440 Hz sine wave with amplitude 0.4 (top) and a 330 Hz square wave with amplitude 0.3 (middle) to produce a new wave (bottom), which is their sum.

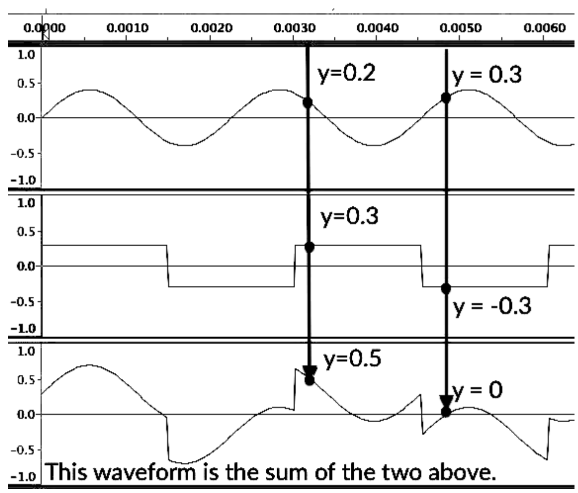


Figure 3.13: *Demonstration of mixing using Audacity. The two top waves are "mixed" (added) to produce the bottom one.*

The Superposition Principle has many implications for acoustics. Its most basic result is that if two *absolutely* identical waveforms are combined, the result is a waveform with double the original amplitude. Moreover, if you add two waveforms that repeat every P seconds, the sum is a waveform that repeats every P seconds. Normally, this means that the combination of two sounds with the same frequency, f, is a sound with frequency f, though it is possible to cook up weird examples where the sum's frequency is some multiple of f, or where the sounds combine to produce silence.

Can two sounds cancel each other out? Yes! Any vibration can be cancelled by an equal and opposite vibration. This is called *destructive interference*. Noise-cancelling headphones use destructive interference. They don't muffle sound—instead, they produce sound with an equal but opposite waveform to the surrounding sound. Adding the two sounds together due to superposition results in sound cancellation.

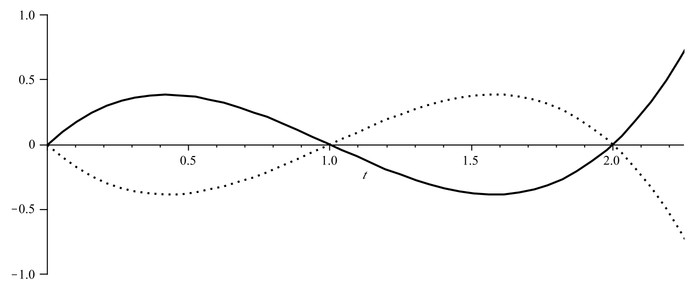


Figure 3.14: *Two functions that cancel each other out. Their sum is zero.*

Mathematically, if a sound is represented by a function $f(t)$, then adding the function $-f(t)$ will cancel the original function (that is, $f(t) + (-f(t)) = 0$). Graphically, the relationship between $f(t)$ and $-f(t)$ is a "flip" around the horizontal axis. In Figure 3.14, $f(t)$ is represented by the solid line and $-f(t)$ is the dotted line.

Demonstration: The Disappearing Tone. Let's use Audacity to demonstrate destructive interference. In Audacity, you can multiply a sound wave by −1 using Effect → Invert.

Produce a tone, then use Tracks → Add New → Audio Track to add a new track. Copy and paste your original tone into the new track so the two tracks are identical. Select a portion of the new track and invert it, as shown in Figure 3.15. If you solo the second track, the inverted portion will sound the same (except you will hear a "pop" where the inversion happens). You will hear pops where the inversion starts and ends.

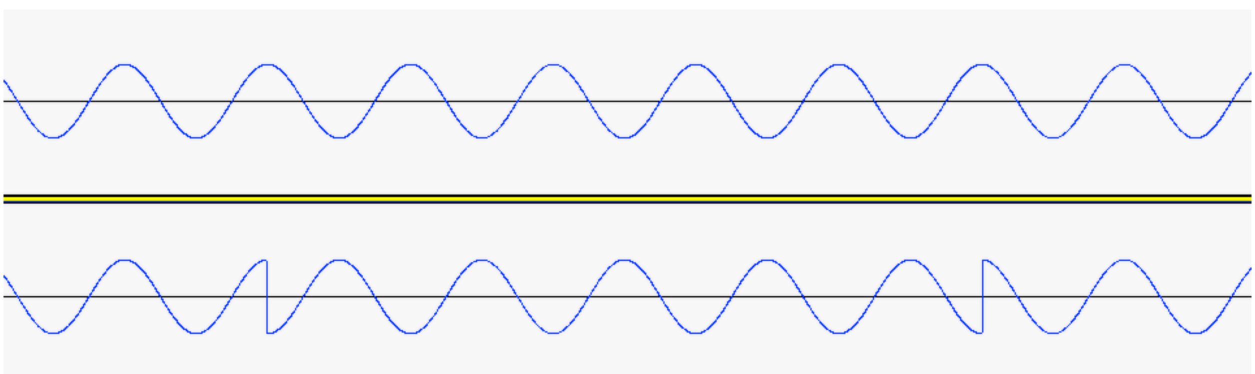


Figure 3.15: *Two sinusoidal waves, with the middle portion of the bottom waveform inverted.*

However, when you play the two tracks together, the sound will disappear at the inversion! Select-All and Tracks → Mix and Render produces the waveform in Figure 3.16. Before the inversion happens, the combined tone is twice the amplitude of the original; during the inversion, the amplitude is zero (silence).

Superposition of sinusoids. Even for sinusoidal waveforms, superposition can produce some unexpected results. Suppose we start with two sinusoids with the same frequency

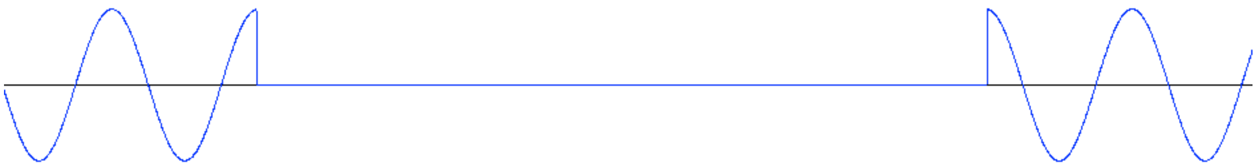


Figure 3.16: *The sum of the waveforms in Figure 3.15*

and phase. In this situation, the amplitude of the sum of the waves equals the sum of the amplitudes of the waves. This phenomenon is called *constructive interference*. For example, if the 50 Hz waveforms $2\sin(100\pi t)$ and $3\sin(100\pi t)$ combine, the result is $5\sin(100\pi t)$, which has amplitude 5 and frequency 50 Hz. Notice that **frequencies do not add**—the frequency of the combination tone is the same as the original tones' frequency.

Now suppose we change the phase of one of the sinusoids.

Demonstration: Phase shift and the secret message. If your original sound is a sinusoid, shifting the wave by half a cycle in the horizontal direction has the same effect as inversion. This corresponds to setting its phase equal to π. Create a stereo track and generate two identical sine waves. Split the track. Zoom in so you can see the waveform clearly and experiment with shifting one tone using the Move button (↔) so it cancels the other one out. Now record a "secret message" and a sine tone in stereo and mix them together. You should be able to split the stereo track and move one track slightly so that the tone disappears and you can hear the secret message.

Exercise 3.10. Generate a 100 Hz sawtooth wave with amplitude 0.5 on Audacity, then choose Tracks → Add New → Audio Track. Select the new track and generate a 1000 Hz sine wave with amplitude 0.5 in that track. You can experiment with the Solo button to hear the tracks separately and together and zoom in to see the waveforms. Then Select All and choose Tracks → Mix and Render. The new sound created is the *sum* of the original waveforms. What does it sound like? What does it look like?

Exercise 3.11. Choose the correct answer and explain why the other answers are incorrect. The Superposition Principle says that if two sound waves reach the same point at the same time, the resulting waveform has (a) a frequency equal to the sum of their frequencies, (b) an amplitude equal to the sum of their amplitudes, (c) a waveform equal to the sum of their waveforms, or (d) all of the above.

Exercise 3.12. In pop music with a single vocalist, the singer's voice is usually recorded equally in both stereo tracks, while the instruments are "panned" to the left or right, meaning that each individual instrument is louder in one track that the other. Explain how the two stereo tracks can be combined to cancel out the vocal without cancelling the instrumental sounds.

3.6 Beats

The phenomenon of "beats" is a direct result of the Superposition Principle. Historically, beats were used to tune keyboard instruments before the advent of electronic tuners. Beats are also important in the theory of dissonance and explain the timbre of some instruments.

Let's perform the following experiment with Audacity:

1. Generate a 440 Hz tone with amplitude 0.4 that lasts for ten seconds. Open a new track and generate a 10 second, 442 Hz tone with amplitude 0.4 that starts about five seconds after the first one.

2. Play each track individually, using the Solo button. Most people—even trained musicians—can't hear a difference between 440 Hz and 442 Hz tones.

3. Listen to both tracks together. You should hear a tone for about five seconds, then a pulsing effect as the two tracks sound simultaneously, and finally a tone that sounds like the first one. The frequency stays (approximately) the same.

4. Select-All and Tracks → Mix and Render to mix the two tracks into one. Figure 3.17 shows the combination track. There appear to be ten pulses, or "beats," where the two tones overlap. Therefore, the frequency of the beats is 2 Hz (that is, 10 cycles per 5 seconds). The maximum amplitude of the envelope that produces the beats is double that of the individual tones.

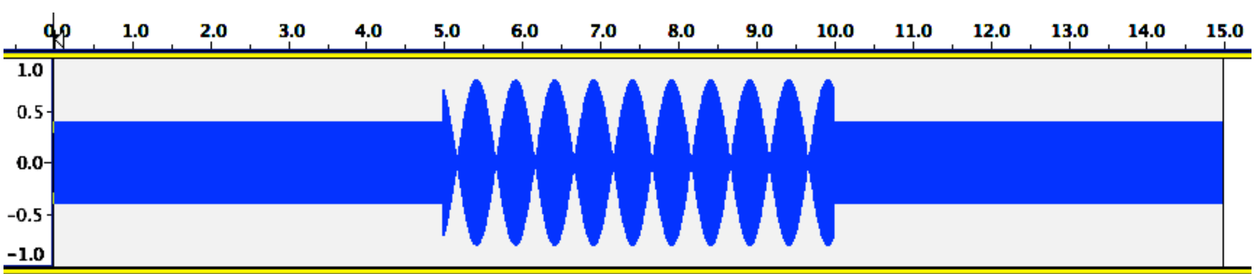


Figure 3.17: *The result of mixing 440 Hz and 442 Hz sinusoids that overlap from the 5 to the 10 second mark.*

This pulsation is an example of *beats* and their frequency is predicted by the Beats Formula. A mathematical derivation of the formula is on page 3–12.

Definition 3.5. Beats *are pulses that are heard when two pitched sounds with close but not equal frequency combine due to superposition.*

Definition 3.6 (The Beats Formula)**.** *Suppose two pitched sounds with close but not equal frequencies, f_1 and f_2, combine and produce beats. Then*

- *The combination sound is pitched and its frequency is the average of the two individual frequencies, $(f_1 + f_2)/2$.*
- *The frequency of the beats is the difference of the two individual frequencies, $|f_1 - f_2|$.*

You might be wondering what "close" means in this context. You are welcome to experiment with Audacity and try different combinations of frequencies. When the tones are less than 20 Hz apart, I hear the combination tone as beats, while I hear two different frequencies when the tones are more than 40 Hz apart. There's a grey area where the combination tone sounds "gritty" without distinctly hearing beats. This is related to the phenomenon of dissonance (literally, bad sound), which we will discuss later in the course. I'll use "less than 20 Hz apart" as the threshold for "close" in the definition.

Example. When pitched sounds of frequency 200 Hz and 204 Hz combine, the result is a 202 Hz tone that pulsates at a rate of 4 Hz. The computations are $(200 + 204)/2 = 202$ (their average) and $|204 - 200| = 4$ (their difference).

Example. Suppose two pitched sounds combine to produce a 1000 Hz tone that pulses at a rate of 7 Hz. What are the frequencies of each sound individually? The answer is $1000 + (7/2) = 1003.5$ Hz and $1000 - (7/2) = 996.5$ Hz.

Application: Why does an accordion sound "bad"? The accordion is especially designed to produce a strident tone that some people don't like (other people, like me, love it!). In the era before amplification, this was a way to make the sound of the instrument cut through other noises and be heard. Each note on the instrument is actually produced by several reeds, each of which sounds at a different frequency. To produce an A note on my accordion, two reeds sound approximately 440 Hz, one sounds 220 Hz, and the fourth sounds 880 Hz. The frequencies 220 Hz, 440 Hz, and 880 Hz form octaves, which sound so similar that a high-voiced and a low-voiced person singing the same melody will naturally sing in octaves. However, the two reeds that are close to 440 Hz are purposely tuned to different frequencies, producing beats. Recordings of my instrument are shown in Figure 3.18. This warbling sound is a characteristic of the accordion—love it or hate it!

Justification for the Beats Formula. The mathematical explanation for beats lies in a formula you might have seen in a high school trigonometry class:

$$\sin a + \sin b = 2 \sin\left(\frac{a+b}{2}\right) \cos\left(\frac{a-b}{2}\right).$$

The sum of the sines of two angles equals twice the sine of the average angle times the cosine of half the difference in the angles.

Now let's suppose f_1 and f_2 are frequencies. For simplicity, let's suppose they both have the same amplitude, A, and zero phase, although a similar formula applies if they don't.

$$A \sin(2\pi f_1 t) + A \sin(2\pi f_2 t) = 2A \sin\left(2\pi \left(\frac{f_1 + f_2}{2}\right) t\right) \cos\left(2\pi \left(\frac{f_1 - f_2}{2}\right) t\right).$$

The combination of sinusoids with frequencies f_1 and f_2 and equal amplitudes is a sinusoid whose frequency is their average frequency times a sinusoid whose frequency is half the

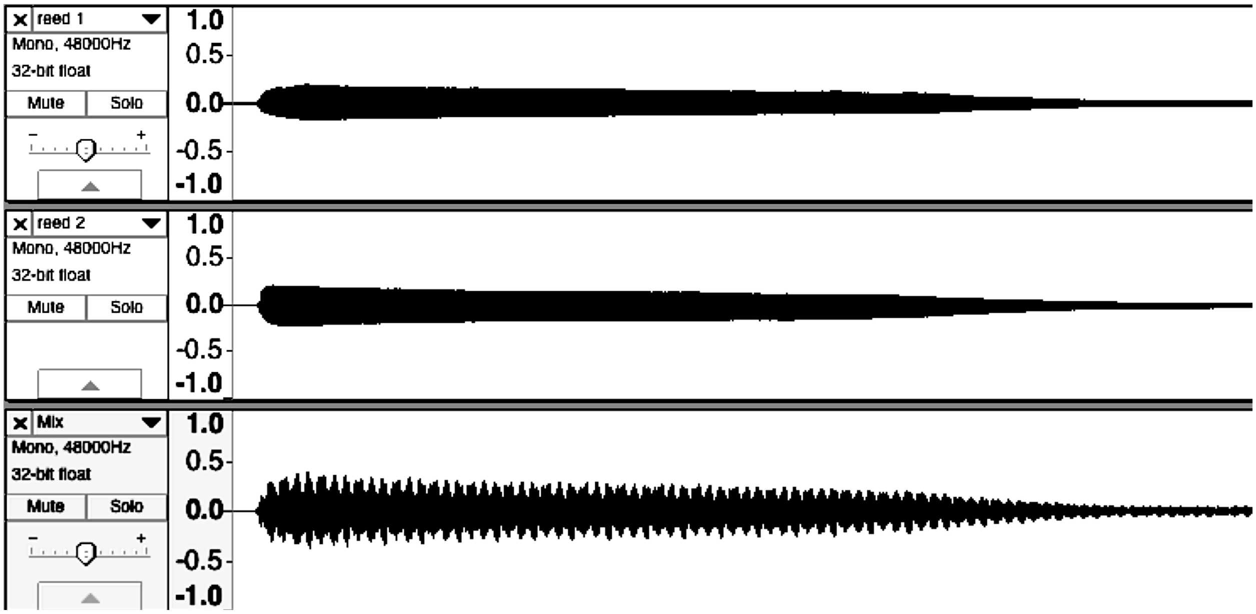


Figure 3.18: *The waveforms of two approximately 440 Hz accordion reeds are shown (top and middle). Beats of about 12 Hz are visible in the mixed waveform (bottom).*

difference in their frequencies. The maximum amplitude of the resulting wave is twice that of the individual sounds. Because its envelope is reflected in the horizontal axis and $\cos x$ is symmetrical around the vertical axis, the frequency of the pulses is two times $|f_1 - f_2|/2$—that is, the absolute difference in frequencies, $|f_1 - f_2|$.

Exercise 3.13. Describe what happens when a 780 Hz tone and a 774 Hz tone combine due to superposition.

Exercise 3.14. Suppose you hear a 300 Hz tone that pulses at a rate of 4 Hz. Describe how this sound could be produced using beats.

Exercise 3.15. An untuned piano often has a different timbre from a tuned piano, even when you play only one note and the correct frequency is heard. What is one reason this might happen? How might a piano tuner correct it? Hint: Look at the inside layout of a piano's strings.

Exercise 3.16. Predict what happens when a 780 Hz tone and a 280 Hz tone combine.

Exercise 3.17. Describe what happens, according to the Superposition Principle, when you are wearing headphones and listen to a 300 Hz tone in your right ear and a 308 Hz tone in your left ear. Use Audacity to produce 300 Hz and 308 Hz tones and adjust the pan (the balance between left and right ears or speakers) so each ear hears a different tone. Listen with headphones. What do you hear? Research *binaural beats* and their relationship to this experiment.

3.7 The wave equation

How do musical instruments produce pitched sounds? Although the mechanism of each type of instrument is different, there is one particular physical equation, the *one-dimensional wave equation*, that applies to stringed and wind instruments. It relates the physical properties of a string or a wind instrument's air column to its patterns of vibration and predicts which frequencies will be sounded.[2]

Vibration of a string

Let's start with a vibrating string. Plucking a string sets it in a back-and-forth motion that causes vibration both in the air and in the body of the instrument, which amplifies its sound. The wave equation says that a vibrating string produces a periodic waveform that is a sum of sinusoids. Moreover, the frequencies of these sinusoids depend not only on the length, tension, and heaviness of the string but also on the shape (called the "harmonic") that the string makes when it vibrates. Here is a summary of the predictions of the wave equation.

Frequency formula for a vibrating string. *A vibrating string produces a periodic waveform that is a sum of sinusoids. The frequency of each sinusoid is*

$$f = \frac{an}{2L}$$

where

- n *is a positive integer that indicates the harmonic of the string*
- L *is the length of the string*
- a *is a number that depends on the tension and material of the string.*

There are three ways to alter the frequency of a string: change n, change L, or change a by changing its tension. An additional way to change a is to use a string with a heavier or lighter material. Each of these is explained in more detail below.

Harmonics of a string (n). The effect produced by changing n in the wave equation can be observed using slow-motion photography. As it vibrates at certain frequencies, the string forms sinusoidal shapes, called *standing waves*, which are the its *modes of vibration*. Figure 3.19 shows the first seven modes of vibration. The string moves back and forth between the two sinusoids that intersect to make each picture. The points at which the string appears stationary are called *nodes*. The two ends of the string are also nodes.

[2]The wave equation is an example of a partial differential equation, meaning that it's a relationship between partial derivatives, which you might have studied in a Calculus class. A mathematical explanation of the wave equation is found in Section **??**.

Between the nodes are *half waves*—individual "humps" of the sinusoid. There must be a whole number of half-waves spanning the entire length of the string, and this number equals n in the frequency formula.

You can force a guitar string to produce a particular standing wave by lightly touching your finger to the string at a point that is a node of that wave, but not a node of any longer wave, and then plucking the string. For example, the third mode of vibration results when a node is created either 1/3 or 2/3 of the way along the string. Guitarists call this technique "playing harmonics."

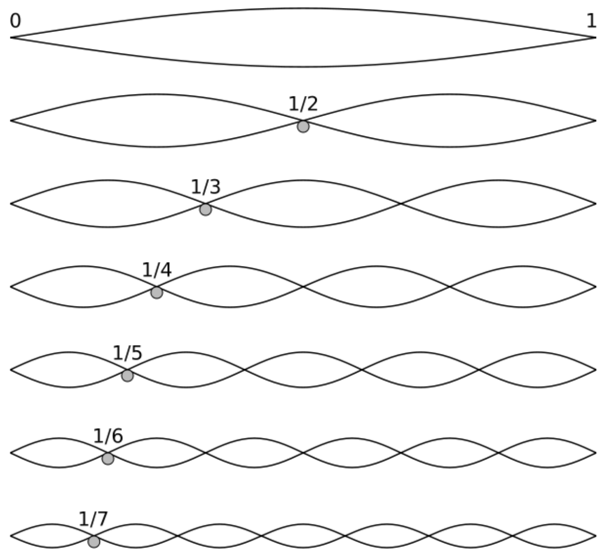


Figure 3.19: *Standing waves for the first seven modes of vibration of a string.*

What is the relationship between standing waves and frequency? Suppose the first mode of vibration has frequency f. Since the first mode has one half-wave, $n = 1$, and therefore $f = a/(2L)$. For the second mode of vibration, $n = 2$ and a and L are the same. So the frequency of the second mode is $2a/(2L)$, which equals $2f$. In general, if the frequency of the first mode of vibration is f, then the second mode has frequency $2f$, the third mode has frequency $3f$, and so on.

Definition 3.7 (Harmonics)**.** *The harmonics of a string are the frequencies $f, 2f, 3f, 4f, \ldots$ which correspond to its modes of vibration. The frequency of the* first harmonic*, f, is called the* fundamental frequency*.*

Harmonics are also called *overtones*, but the numbering is different: the second harmonic is the first overtone, and so on.

When we don't force it to vibrate in a particular mode, the wave equation predicts that a string actually vibrates in a sum of the waveforms of its harmonics. However, the fundamental frequency is normally the strongest—the component of the sound that has the largest amplitude—while the higher harmonics are quieter. For this reason, we tend to perceive the higher harmonics as part of the quality of the sound rather than as individual tones. However, overtone singers are able to amplify certain harmonics to the extent that we do hear them as separate tones.

Example. Suppose a string vibrates with a fundamental frequency of 300 Hz. The frequencies of its first four harmonics are 300 Hz, 600 Hz, 900 Hz, and 1200 Hz.

Length of a string (L). Harmonics only partly explain the frequencies that a stringed instrument can produce. The most common way that a player makes a different frequency is to press the string down with a finger of their left hand so it can't vibrate at that point. A guitar has frets—little metal bars along the neck of the instrument—to make this easier. When the string is pressed onto a fret, the fret acts like a new end of the string. The effect would be the same if the string were cut shorter and fixed at both ends with the same tension.

Example. The fundamental frequency of a violin's A string is 440 Hz and the length of the part of the string that vibrates is 327 mm. Then $440 = a/(2 \cdot 327)$, so $a = 440 \cdot 2 \cdot 327$. Suppose a player then presses the string down near its end, so that only 291 mm of the string can vibrate. The fundamental frequency produced is

$$f = \frac{an}{2L} = \frac{440 \cdot 2 \cdot 327}{2 \cdot 291} \approx 494 \text{ Hz}.$$

Note that the fundamental frequency of the shortened string is found by multiplying 440 Hz by the reciprocal of the ratio of lengths, 291/327. Therefore, $\frac{291}{327} \approx \frac{440}{494}$.

In fact, since the value of a cancels out, this relationship is always true, and gives us a handy formula:

Length-frequency ratio formula. *If the length of a string is changed without changing its tension, the ratio of lengths equals the reciprocal of the ratio of fundamental frequencies. That is,*

$$\frac{L_1}{L_2} = \frac{f_2}{f_1},$$

where the original string has length L_1 and fundamental frequency f_1 and the altered string has length L_2 and fundamental frequency f_2.

Example. The fundamental frequency of a cello's A string is 220 Hz. Where would the player need to press down to make the string vibrate at 330 Hz? In the formula, use $f_1 = 220$ and $f_2 = 330$. Since the ratio of frequencies is $330/220 = 3/2$, the ratio of lengths is $2/3$. The player presses down at the point two-thirds of the distance from the top end of the string. (Note that we don't need to know the string's length to make this computation.)

Tension and mass of a string (a). In addition to altering a string's length or mode of vibration, you can change its tension or choose a string made of heavier or lighter material. These last two choices affect the value of a, which represents the speed at which a wave travels along the stretched string. If T is the tension of the string and m is its mass per unit length, then $a = \sqrt{T/m}$. Therefore, assuming that L and n don't change, increasing the tension results in higher frequency, while increasing the mass of the string results in a lower frequency.

Stringed instruments have *tuning pegs* that the player turns to change the string's tension. However, there is only so much that tuning pegs can do—strings can break if they're stretched too tight, while they flop if they're not tight enough. To avoid this, a lighter string will be used for a higher sound, so that you change the mass per unit length, m, in the formula for a. This is why instrument strings are not interchangeable: strings intended for lower sounds are weightier than those intended for higher sounds.

Wind instruments

In a wind or brass instrument, standing waves are bands of low and high pressure; the pressure at an open end of the instrument is held constant, creating a node, just like the ends of the string are fixed by the bridges. Though the physics of wind instruments is different from those of stringed instruments, it turns out that the one-dimensional wave equation still holds, and (most) wind instruments have standing waves that divide the instrument into a whole number of half-waves. The frequencies produced are still $an/(2L)$, but a depends on the shape of the instrument and air pressure.

There is some more complicated behavior that explains the clarinet and a few other instruments. In these instruments, there is a node at one end of the air column and an antinode—the midpoint between two nodes—at the other. The frequencies produced are $an/(4L)$, where n is a positive odd integer. If you play both the clarinet and saxophone, for example, you might have noticed that opening the register key takes you up an octave for the saxophone and an octave plus a fifth for the clarinet. This is because the clarinet's second mode of vibration has a frequency 3 times the fundamental frequency.

The two- and three-dimensional wave equations

Modes of vibration and harmonics are the same thing only if an instrument obeys the one-dimensional wave equation. Some instruments have non-harmonic modes of vibration that reflect their two- or three-dimensional geometry, as with timpani, bells, and singing bowls, which are members of the bell family. The wave equation that we used for stringed and wind instruments is called the one-dimensional wave equation because those instruments are essentially "one-dimensional"—meaning that we only have one position variable, x, in the equation. However, in a drum, you have two position variables (x and y) because the drum head is two-dimensional, while in a bell you need three position variables (x, y, and z). The *two-* and *three-dimensional wave equations* are needed for drums and bells. The solutions to these equations do *not* result in harmonics.

One way to understand this difference visually is to think of the standing waves that are possible in a string. We know these result in frequencies that are multiples of the fundamental. However, a drum vibrates in a different way, and its modes of vibration are much more complex. There are several excellent demonstrations on YouTube. Try

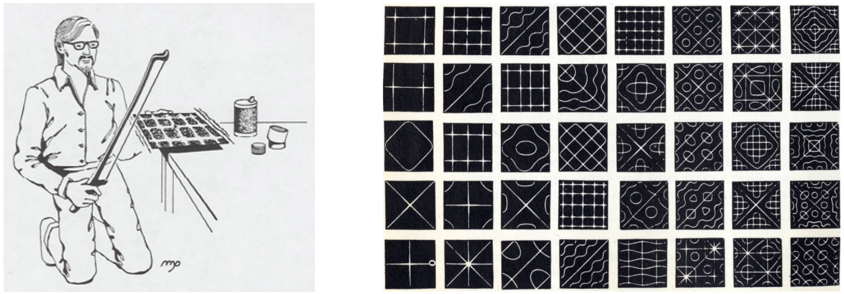


Figure 3.20: *Chladni plate experiment from the UCLA Physics Lab demonstrations web site and Chladni patterns published by John Tyndall in 1869.*

searching for “drum modes vibration” and “Chladni plates.” *Chladni plates* are thin metal plates that can be vibrated, either by bowing or electronically (see Figure 3.20). Different frequencies and shapes produce different standing waves whose nodes are shown as white curves and lines in Figure 3.20 (right). The Wikipedia page “Vibrations of a circular drum” has computer-generated animations.

Exercise 3.18. Suppose a string vibrates with a fundamental frequency of 200 Hz. Find the frequencies of the first four harmonics, and sketch the standing waves corresponding to those harmonics.

Exercise 3.19. Explain the difference in frequency between pressing the string down 1/3 of the way along its length and lightly touching it at the same point.

Exercise 3.20. Suppose the fundamental frequency of a string is 50 Hz. List a few of its harmonics. What is the minimum frequency difference between harmonics? Can harmonics combine with each other due to superposition and produce beats? Is it possible for beats to be produced by harmonics, in general?

3.8 Solutions to exercises

Solution 3.1. The period of the cello sound is obtained by zooming in on the waveform and locating a cycle. The length of the cycle is the difference between its starting and ending times. For example, the cycle that starts at 3.570 seconds ends at 3.580 seconds. The length of the cycle is $3.580 - 3.570 = 0.01$ seconds, which is the period. (The correct units for this measurement are seconds.)

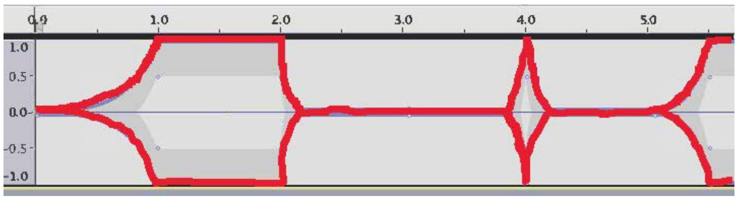
Solution 3.2. In a previous exercise, we calculated that the period of the cello sound is approximately 0.0102 seconds. The frequency is approximately $1/0.0102$ cycles per second $\approx$ 98.04 Hertz. It is most like the second tone.

Solution 3.3. The period of the lowest audible sound is $1/20 = 0.05$ seconds. The period of the highest audible sound is $1/20000 = 0.00005$ seconds.

Solution 3.4. The frequency of the violin's sound is $1/0.002 = 500$ Hz. The frequency of the clarinet's sound is $1/0.0025 = 400$ Hz. The clarinet's sound is lower because its frequency is smaller.

Solution 3.5. Using Change Pitch to double the frequency of the pitched sound results in a higher sound. It doesn't seem to have an effect on the white noise.

Solution 3.6. Here is the envelope:

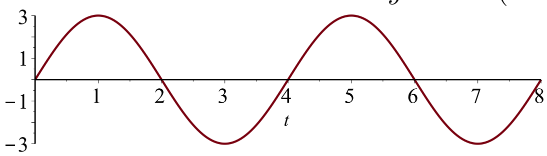


You would hear a 440 Hz tone that fades in for the first second, fades out rapidly after second 2, is silent from about second 2.2 to second 3.6, has a short burst of volume at second 4, then fades in after second 5.

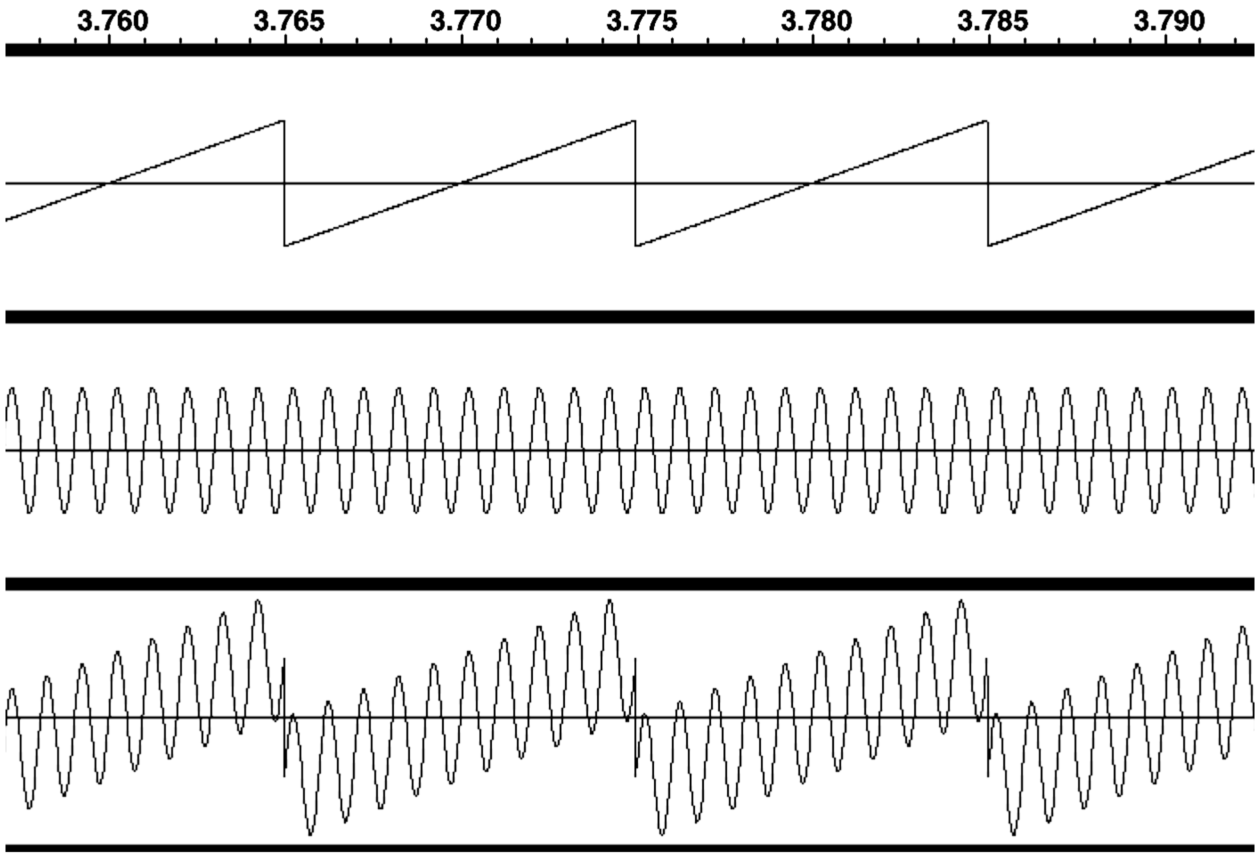
Solution 3.7. In this formula, $A = 0.4$ is the amplitude, the frequency is $100\pi/(2\pi) = 50$ Hz, and the period is the reciprocal of the frequency, $1/50 = 0.02$ seconds.

Solution 3.8. Amplitude $= 1 = A$ and period $= 0.5$ seconds, so $F = 2$ and the formula is $y = \sin(4\pi t)$.

Solution 3.9. The formula is $y = 3\sin(2\pi t/4) = 3\sin(\pi t/2)$.



Solution 3.10. The original tracks and the mix track (the sum) are shown. The 100 Hz sawtooth wave is a low buzzing sound. The 1000 Hz sine wave is a high, pure tone. The combination of the two produces a buzzing sound that mostly sounds like the 100 Hz tone. It's hard to hear the high tone in the mix.



Solution 3.11. It follows from the definition of the Superposition Principle that (c) is true. An example that shows that (a) is false is constructive interference: when two sinusoids with the same phase and frequency combine, the resulting wave has a greater amplitude but the *same* frequency. The phenomenon of destructive interference shows that (b) is false. In that case, two waves can combine to produce silence (amplitude = 0). Therefore, the correct answer is (c).

Solution 3.12. The trick is to separate the stereo tracks, invert one track, then mix the two tracks together. Any waveform that was equally present in both left and right tracks will be cancelled out.

Solution 3.13. The result is a 777 Hz tone that pulses at a rate of 6 Hz (equivalently, you hear 6 beats per second). The maximum amplitude of the beats is 0.6.

Solution 3.14. This sound is produced when a 298 Hz tone and a 302 Hz tone combine due to superposition.

Solution 3.15. Most notes on the piano are made when two or three strings of identical frequencies are hit. When these strings are slightly out of tune with each other, beats create a pulsing sound which is particularly unpleasant when the beats are fast. For example, suppose you strike the A below middle C on a badly tuned piano and hear a 220 Hz tone that pulses at a rate of 10 Hz. It is produced by two strings whose average frequency is 220 Hz. The difference in their frequencies is 10 Hz. Therefore, their frequencies are $220 + (10/2) = 225$ Hz and $220 - (10/2) = 215$ Hz. The piano tuner must tune both strings to 220 Hz and the beats will disappear.

Solution 3.16. You would not hear beats or pulses because the frequencies are too far apart. Instead, you perceive two different frequencies. As a rule of thumb, only tones less than 20 Hz apart produce beats.

Solution 3.17. You would not hear beats because superposition does not occur. You would hear two separate tones. They are so close in frequency that you would probably perceive them as the same pitch. The "theory" of binaural beats states that your brain is able to combine the sounds and mentally perceive beats if you try hard enough—or at least, the beats have some effect that may or may not get you high, cure cancer, etc. There is no scientific evidence for binaural beats.

Solution 3.18. The first four harmonics are 200 Hz, 400 Hz, 600 Hz, and 800 Hz. Note that the first harmonic is always the same as the fundamental frequency. The standing waves look like the first four waves in the picture on page 3–15.

Solution 3.19. If you press the string down, you are creating a string that is effectively shorter. The fundamental frequency of the longer $2/3$ part of the string equals $3/2$ times the original fundamental frequency. If you lightly touch the string, you are creating a node and forcing the string to sound its third harmonic, which has a frequency 3 times the original fundamental frequency and an octave higher than the frequency made by pressing down.

Solution 3.20. The harmonics are 50 Hz, 100 Hz, 150 Hz, etc. Since the smallest distance between harmonics is 50 Hz, beats cannot be produced. In general, the distance between successive harmonics equals the fundamental frequency. Since the fundamental frequency of an audible sound is 20 Hz at a minimum, the harmonics can never be close enough in frequency to produce beats.