Sounding Number
Dr. Rachel Hall

Intervals and Pitch

A musical *interval* is a relationship between two frequencies. The most fundamental musical intervals are the *octave*, which corresponds to a 2:1 frequency ratio, and the *perfect fifth*, which corresponds to a 3:2 frequency ratio. Suppose we start with a frequency f. Then

$(1/2)f =$	an octave below f	$(2/3)f =$	a fifth below f
$2f =$	an octave above f	$(3/2)f =$	a fifth above f
$4f =$	two octaves above f	$(9/4)f =$	two fifths above f
$8f =$	three octaves above f	$(27/8)f =$	three fifths above f

Notice that $2 = 2^1$, $4 = 2^2$, $8 = 2^3$, and $(1/2) = 2^{-1}$. So we have the formulas:

$$2^n f = \quad n \text{ octaves above } f \qquad\qquad (3/2)^n f = \quad n \text{ fifths above } f$$

This formula makes sense for any whole number n if we interpret a *negative* value of n to mean "octaves below."[1] So (-3) octaves above f is the same as 3 octaves below f, and the frequency is $2^{-3}f = (1/2^3)f = (1/8)f$.

We can combine octaves and fifths to form new intervals. For example, if two frequencies are in a 3:1 ratio, then the interval between them is an octave plus a fifth. Let's take this step-by-step: the frequency an octave above f is $2f$; an fifth above $2f$ is $(3/2)(2f) = 3f$. Notice that it doesn't matter whether we go up a fifth, then an octave, or vice versa. This gives the formulas

> The frequency n octaves plus m fifths above f is $2^n(3/2)^m f$.
> The frequency n octaves plus m fifths below f is $2^{-n}(3/2)^{-m} f$.
> Negative values of n or m are interpreted to mean octaves or fifths below.

Examples.

1. The frequency two octaves plus one fifth above 900 Hz is found by setting $n = 2$ and $m = 1$, so the answer is $2^2(3/2)^1 900 = 5400$ Hz.

2. The frequency two octaves plus one fifth below 900 Hz is found by setting $n = -2$ and $m = -1$, so the answer is $2^{-2}(3/2)^{-1}900 = (1/4)(2/3)900 = 150$ Hz.

3. The frequency two octaves minus three fifths above 900 Hz is found by setting $n = 2$ and $m = -3$, so the answer is $2^2(3/2)^{-3}900 = (4)(2/3)^3 900 = 1066\frac{2}{3}$ Hz.

4. The frequency two fifths minus one octave above 900 Hz is found by setting $n = -1$ and $m = 2$, so the answer is $2^{-1}(3/2)^2 900 = (1/2)(3/2)^2 900 = 1012\frac{1}{2}$ Hz.

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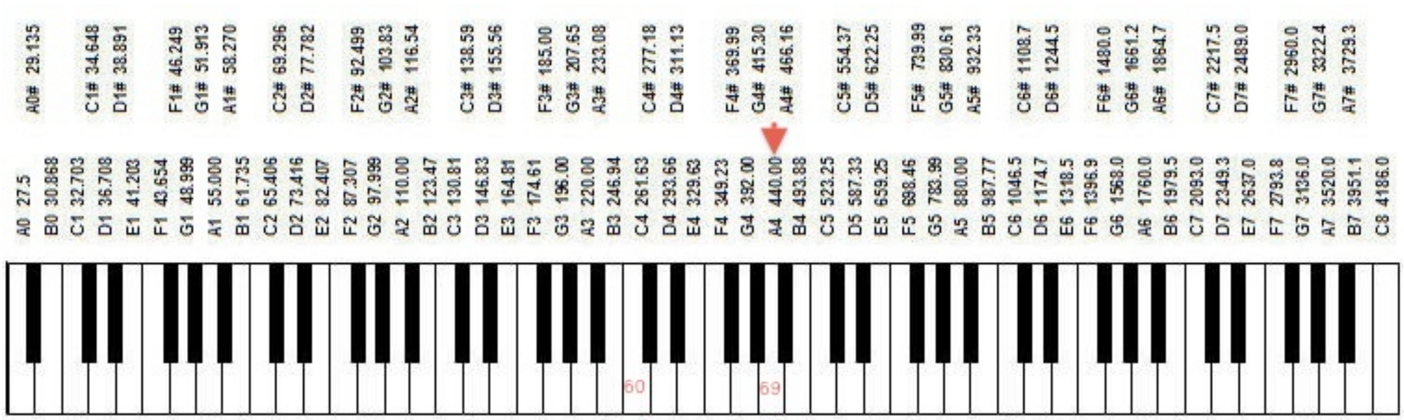
[1] Negative exponents indicate division: $a^{-n} = \frac{1}{a^n}$. A zero exponent results in a value of 1; that is, $a^0 = 1$.

EXERCISE 1. Find m and n and calculate the following frequencies:

1. Two octaves plus three fifths above 440 Hz.
2. Two octaves plus three fifths below 440 Hz.
3. One octave minus one fifth above 440 Hz.
4. Twelve fifths above 100 Hz.
5. Seven octaves above 100 Hz.
6. Twelve fifths minus seven octaves above 100 Hz (to two decimal places). What would you hear if you played this frequency and 100 Hz simultaneously?

Pitch Our ears don't judge distance in the same way that we measure frequency. For example, we hear the distance between any two notes an octave apart as the same—so we hear the distance from 110 Hz to 220 Hz as being the same as the distance from 440 Hz to 880 Hz. Therefore, we use the system of *pitch* to measure frequency in a way that corresponds to human perception. Twelve units of pitch equals one octave. There are two ways to notate pitch: either using a note name (a letter from A-G) and a number to indicate the octave, or using MIDI notation, in which each key on the piano corresponds to a whole number from 21 to 108.

Here is a picture of a piano keyboard with the frequencies and pitches labeled. In addition, two MIDI numbers are shown: 69 (A4) and 60 (C4, or "middle C").



EXERCISE 2. Circle all the piano keys producing pitches that are a whole number of octaves above or below 440 Hz. This pitch is called "A4." Write the note names of the pitches you circled. (Each note name is a letter plus a number.)

EXERCISE 3. Find the MIDI numbers of the piano keys you circled.

EXERCISE 4. Find the frequencies that are one or two fifths above or below 440 Hz. Locate their best approximations on the keyboard and put an "X" on these keys.

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Pitch and Frequency

Converting between frequency and pitch. Let's make a table of frequencies and pitches, using MIDI numbers:

	frequency	pitch	
$440 \cdot 2^{-4}$	27.5 Hz	21	$69 - 4 \cdot 12$
$440 \cdot 2^{-3}$	55 Hz	33	$69 - 3 \cdot 12$
$440 \cdot 2^{-2}$	110 Hz	45	$69 - 2 \cdot 12$
$440 \cdot 2^{-1}$	220 Hz	57	$69 - 1 \cdot 12$
$440 \cdot 2^{0}$	440 Hz	69	$69 + 0 \cdot 12$
$440 \cdot 2^{1}$	880 Hz	81	$69 + 1 \cdot 12$
$440 \cdot 2^{2}$	1760 Hz	93	$69 + 2 \cdot 12$
$440 \cdot 2^{3}$	3520 Hz	105	$69 + 3 \cdot 12$

This conversion table suggests that the frequency $440 \cdot 2^x$ Hz corresponds to MIDI pitch number $69 + 12x$. In fact, this relationship is valid even if x is not a whole number. For example, let's calculate the frequency for pitch 60 (middle C). First, we find the value of x. Since pitch $= 60 = 69 + 12x$, $x = -9/12 = -3/4$. Then

$$\text{frequency } = 440 \cdot 2^x = 440 \cdot 2^{-3/4} \approx 261.63 \text{ Hz}.$$

Checking on the piano chart, we see that the frequency of middle C is indeed 261.63 Hz.

The general pitch-to-frequency conversion formula is

$$\boxed{\text{the frequency corresponding to pitch } p \text{ is } f = 440 \cdot 2^{(p-69)/12} \text{ Hz.}}$$

In order to convert from frequency to pitch, we need a way to write any given frequency f in the form $440 \cdot 2^x$ (that is, we need to find x in terms of f). Since $f/440 = 2^x$, x is the exponent of 2 that produces $(f/440)$. This exponent is called the *logarithm base 2 of* $(f/440)$ and written

$$x = \log_2(f/440).$$

The general frequency-to-pitch conversion formula is

$$\boxed{\text{the pitch corresponding to frequency } f \text{ is } p = 69 + 12\log_2(f/440).}$$

Note: most calculators don't compute $\log_2$ directly, so you have to use the fact that $\log_2 x = \log x/\log 2$.

Example. Approximate the pitch corresponding to 660 Hz to two decimal places.

$$\begin{aligned} p &= 69 + 12\log_2(660/440) \\ &= 69 + 12\log(3/2)/\log(2) \approx 76.02 \end{aligned}$$

EXERCISE 5. Approximate to two decimal places

1. the frequency of pitch 72
2. the pitch of 1000 Hz
3. the frequency of pitch 1
4. the pitch of the lowest hearable frequency
5. the pitch of the highest hearable frequency
6. the pitch that is two fifths above 440 Hz

Intervals in pitch. We've defined intervals such as octaves and fifths as being *ratios* of frequencies. When we measure pitch, intervals are *differences* in pitch. For example, if a frequency corresponds to pitch p, then the frequency an octave above it corresponds to pitch $(12 + p)$. Let's calculate the difference in pitch between frequencies f and rf (think of r as the ratio):

$$(69 + 12 \log_2(rf/440)) - (69 + 12 \log_2(f/440)) = 12(\log_2(rf/440) - \log_2(f/440)).$$

To simplify this expression, we need to use the fact that if A and B are positive numbers, $\log(A \cdot B) = \log A + \log B$ and $\log(A/B) = \log A - \log B$. Therefore,

$$\boxed{\text{The interval in pitch from frequency } f \text{ to frequency } rf \text{ is } 12 \log_2(r).}$$

For example, an octave, measured in pitch, is $12 \log_2(2) = 12$. What is the pitch value of a fifth?

Answers to Exercises.

EXERCISE 1.

1. $n = 2$; $m = 3$; frequency = $2^2(3/2)^3 440 = 5940$ Hz.
2. $n = 2$; $m = 3$; frequency = $2^{-2}(3/2)^{-3} 440 = 32.59$ Hz.
3. $n = 1$; $m = -1$; frequency = $2^1(3/2)^{-1} 440 = 586\frac{2}{3}$ Hz.
4. $n = 0$; $m = 12$; frequency = $2^0(3/2)^{12} 100 = 12975$ Hz.
5. $n = 7$; $m = 0$; frequency = $2^7(3/2)^0 100 = 12800$ Hz.
6. $n = -7$; $m = 12$; frequency = $2^{-7}(3/2)^1 2100 = 101.36$ Hz. You would hear beats at a rate of 1.36 Hz.

EXERCISE 5.

1. 523.25 Hz
2. 83.21
3. 8.66 Hz
4. 15.49
5. 135.08
6. $69 + 12 \log_2(9/4) \approx 83.04$