Chapter 2

Math for Poets

> . . .
> But most by Numbers judge a Poet's Song,
> And smooth or rough, with them, is right or wrong;
> . . .
> These Equal Syllables alone require,
> Tho' oft the Ear the open Vowels tire,
> While Expletives their feeble Aid do join,
> And ten low Words oft creep in one dull Line,
> . . .
>
> —Alexander Pope, *An Essay on Criticism* (1709)

Numerical patterns have fascinated humans for millennia: numbers that are powers of other numbers, squares that are sums of squares, numbers that form intriguing lists. This is the story of one of the earliest studies of rhythm, an investigation that led ancient Indian scholars to discover the mathematical patterns that Westerners know as the Fibonacci numbers, Pascal's triangle, and the binary counting system. Although our story initially concerns rhythm in poetry, the Ancient Indians' ability and fascination with exploring rhythmic patterns also had a profound influence on their music.

2.1 Meter as binary pattern

In English, a poetic rhythm, called a *meter*, is a pattern of stressed and unstressed syllables. English poets use about a dozen different meters. Much poetry, including Shakespeare's plays, is written in *iambic pentameter*—five pairs of alternating unstressed and stressed syllables to a line. Alexander Pope's 700+ line iambic pentameter poem *An Essay on Criticism* (1709) is a good example. In the excerpt that begins this chapter, he ridicules critics who judge poetry "by numbers"—that is, solely on how well a poet follows strict metrical rules.

While English poets use relatively few meters, there are *hundreds* of different meters in Sanskrit, the classical language of India. Syllables in Sanskrit poetry are classified by duration (short or long) rather than stress. Any Sanskrit meter can be written as a *binary pattern*—a pattern of any length formed by two symbols. For example, there are eight binary patterns of length 3 that are formed from the letters L and S: LLL, SLL, LSL, SSL, LLS, SLS, LSS, and SSS. These correspond with the eight three-syllable meters, using S for a short syllable and L for a long syllable.

Pingala is thought to be the first Indian scholar to study meter mathematically. He probably lived in the last few centuries BC. As is typical in ancient Indian literature, Pingala's writings took the form of short, cryptic verses, or *sūtras*, which served as memory aids for a larger set of concepts passed on orally. We are dependent on medieval commentators for transmission and interpretation of Pingala's writings.

Here are two of the questions that Pingala solved:

1. What is a reliable way to list all the meters with a given number of syllables?
2. How many meters have a given number of syllables?

Problem 1: listing meters. There are a number of ways to solve this problem. Pingala's solution would result in the one-syllable meters listed as

L
S

the two-syllable meters being listed as

LL
SL
LS
SS

and the three-syllable meters like this:

LLL
SLL
LSL
SSL
LLS
SLS
LSS
SSS

Here is how the four-syllable patterns would be listed:

LLLL
SLLL
LSLL
SSLL
LLSL
SLSL
LSSL
SSSL
LLLS
SLLS
LSLS
SSLS
LLSS
SLSS
LSSS
SSSS

Several patterns are observable in these lists. The first column alternates L and S, the second alternates pairs of L's and pairs of S's, the third alternates triples of the letters, and so on. There is symmetry in the lists, in the sense that the first pattern is equivalent to the last with the letters exchanged, and this is true for each pair of patterns that are at the same distance from the beginning and end. In addition, there is a relationship between successive lists: for example, the list of four-syllable meters is formed from the list of three-syllable meters by first adding L's to the end of the list, then adding S's. This last observation is useful for describing an algorithm that will produce all meters of length n in the order that Pingala did.

Exercise 1. Write the list of five-syllable meters.

Theorem 1 (Listing n-syllable meters). *Suppose the list of one-syllable patterns is* $\begin{array}{c}L\\S\end{array}$*, and the list of n-syllable patterns is formed from the list of $(n-1)$-syllable patterns by adding L's to the end of each $(n-1)$-syllable pattern, followed by the list resulting from adding S's to the end of each $(n-1)$-syllable pattern. Then each n-syllable pattern will occur exactly once in the list.*

Proof. It is clear that each of the one-syllable meters is listed once. Suppose that the algorithm results in each of the $(n-1)$-syllable patterns being listed exactly once. Choose any n-syllable pattern. Use the algorithm to form a list of n-syllable patterns. We want to show that your chosen pattern occurs exactly once in the new list. If the pattern ends in an L, then the algorithm shows that it appears exactly once in the first half of the list, because the pattern of its first $(n-1)$ syllables appear exactly once in the list of $(n-1)$-syllable patterns. If it ends in an S, it appears exactly once in the second half of the list for the same reason. □

The proof essentially says that if the one-syllable patterns are correct, then the two-syllable patterns are correct, then the three-syllable patterns are correct, and so on, until your chosen length is reached—sort of like a row of dominoes falling down. This type of reasoning is called *proof by induction.* Although it may seem reasonable to argue this way, in fact, the *axiom of induction* is required to allow it.

Problem 2: counting meters. How many meters have n syllables? Counting the patterns on the lists I have typed, you see the numbers 2, 4, and 8, which are equal to 2^1, 2^2, and 2^3. You might conjecture that there are 16 (or 2^4) four-syllable meters, and, in general, there are 2^n n-syllable meters. This is correct. It follows so closely from the theorem that mathematicians would called it a *corollary*, which is a theorem that may be proven from another theorem without much effort.

Corollary 1 (Counting n-syllable meters). *The number of n-syllable meters is 2^n.*

Exercise 2. How long is the list on which LLSSLSL appears?

Exercise 3. Assuming that the theorem has been proven true, explain why the corollary is true. You don't need to write a formal proof.

The binary number system Since there's nothing special about the letters L and S, the previous theorem and its corollary generalize to any set of binary patterns. For example, there are 2^5 patterns of length 5 that are formed from the letters a and b. In some ways, Pingala anticipated the development of the binary number system. The binary number system is a base-two positional number system (our number system is a base-ten positional system). It has two digits, 0 and 1, and its place values are powers of two—therefore, every number is also a binary pattern. The decimal numbers 1, 2, 8, and 11 have binary representation 1, 10, 1000, and 1011, respectively. The binary number system was not fully described until Gottfried Leibniz did so in the seventeenth century.

Exercise 4. Suppose you flip a coin three times and write down the pattern of heads and tails, using H and T. The order of flips makes a difference—that is HHT is different from HTH. How many

1 beat	2 beats	3 beats	4 beats	5 beats
S	L SS	SL LS	LL SLS	SLL LLS
		SSS	SSL LSS	LSL SSLS
			SSSS	SSSL SLSS
				LSSS
				SSSSS

Figure 2.1: Meters listed by duration

patterns of three coin flips are there? List them and use your list to compute the likelihood of getting tails exactly once if you flip three times. Describe how you would list and count the patterns for any number of coin flips.

Exercise 5. Here are the binary numbers from 8 (1000 in binary) to 15 (1111 in binary):

1000
1001
1010
1011
1100
1101
1110
1111

How is this sequence related to the list of three-syllable meters? Conjecture how many binary numbers have five digits and list them. (Extra Credit: Learn more about the binary number system and determine whether your answer is correct.)

2.2 The Hemachandra-Fibonacci numbers

The 12th-century writer Ācārya Hemachandra also studied poetic meter. A *mora* is the durational unit of Sanskrit poetry; short syllables count as one mora and long syllables two morae, which we'll call "beats." Instead of counting meters with a fixed number of syllables, Hemachandra counted meters having a fixed duration. For example, here are the three meters of three beats: SL, LS, and SSS. More meters are listed in Figure 2.1.

Exercise 6. Before you go on, count the number of meters for duration one through five and make a conjecture about the number of meters with six beats and the formula for finding the number of meters with any arbitrary number of beats.

Hemachandra noticed that each number in the sequence is the sum of the two previous numbers. Since the first two numbers are 1 and 2, the numbers form the sequence $1, 2, 3, 5, 8, 13 \ldots$. In other words, he discovered the "Fibonacci" numbers—about fifty years before Fibonacci did (the Fibonacci sequence starts with $1, 1, 2, 3, 5, \ldots$, but otherwise follows the same pattern).[1] Indian poets and drummers know these numbers as "Hemachandra numbers."

[1]Fibonacci may have learned the sequence from the Indians. Fibonacci was educated in North Africa and was familiar with Eastern mathematics. His *Liber Abaci* (1202), in which the sequence appears, introduced

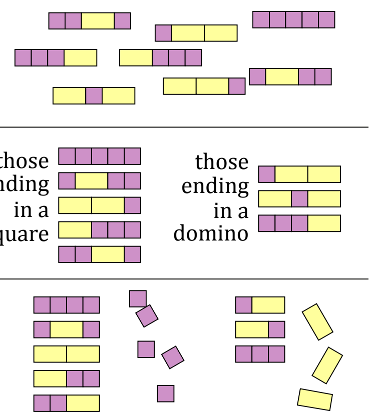


Figure 2.2: Hemachandra's problem is equivalent to the "domino-square problem": in how many ways can 1×2 dominoes and 1×1 squares tile a $1 \times n$ rectangle? Here is a visual demonstration that $H_n = H_{n-2} + H_{n-1}$.

Theorem 2. *The sequence of numbers of meters with n beats, beginning with $n = 1$, is the Hemachandra sequence, $1, 2, 3, 5, 8, 13, \ldots$, where each following number is the sum of the two previous.*

Proof. Suppose H_n represents the total number of patterns of duration n. Partition the collection of n-beat patterns into two sets: patterns of duration $n-2$ followed by a long syllable and patterns of duration $n-1$ followed by a short syllable. The number of patterns in the first set equals H_{n-2}, since they are formed by adding L to the patterns with $n-2$ beats, and the number of patterns in the second set equals H_{n-1}, since they are formed by adding S to the patterns with $n-1$ beats. The partition shows that $H_n = H_{n-2} + H_{n-1}$. Since $H_1 = 1$ and $H_2 = 2$, we obtain the Hemachandra sequence. □

Figure 2.2 gives a visual demonstration in which short and long syllables are represented by squares and dominoes, respectively.

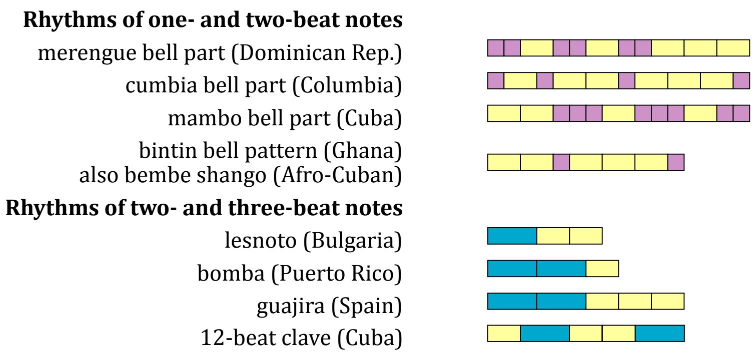
Exercise 7. The algorithm suggests how use the information in figure 2.1 to list the 13 patterns with six beats. Write them.

Recursion. *Recursion* is a process in which one structure is embedded inside another similar structure, rather like nesting Russian dolls, or the Droste cocoa box. Recursion is the lifeblood of computer programming, and are important in mathematics as well. An algorithm is a *recursive* if you start with some information (called a *base case*) and arrive at all subsequent information by repeatedly applying the same rule, called a *recursive rule*.

In the example of the Hemachandra numbers, there are two meters of one syllable each, L and S. This is the *base case*. If you know all the meters that have n syllables, you can list the meters that have $n+1$ syllables by first adding an L to the beginning of the meters with n syllables, then adding an S to the beginning of the meters with n syllables. This is the *recursive rule*.

the Indian positional number system—the system we use today—to the West. However, his description of the number sequence as counting the sizes of successive generations of rabbits is not found in India.

The Padovan sequence. The poetic meters that Pingala and Hemachandra studied have an analogue in music. Music from India, the Middle East, and the Balkans is often written in *additive meter*—that is, a rhythmic organization founded in grouping beats rather than subdividing larger units of time called measures (the typical metrical structure of Western European music). For example, the Bulgarian dance called *Daichovo horo* has a nine-beat measure, grouped 2+2+2+3. This means that the first, third, fifth, and seventh beats normally receive an accent; they are also the beats on which the dancers step. The jazz pianist and composer Dave Brubeck (1920-2012) used the same rhythm in his "Blue Rondo à la Turk" (1959). A *Gankino horo* has an eleven-beat measure, with beats grouped 2+2+3+2+2.



Many additive meters are binary patterns formed of two- and three-beat groupings. Pingala's algorithm will list all additive meters formed of n groups of beats for any n. However, musical patterns are typically classified by their number of beats. In this situation, we need something like Hemachandra's sequence for counting meters of a given duration, as explored in the following exercise:

Exercise 8. The Hemachandra numbers count meters formed from one- and two-beat groups. What sequence counts meters consisting of two- and three-beat groups? Find the first few numbers in the sequence—the base case—and a recursive rule that generates the sequence. Explain why your rule is correct. This number sequence is called the *Padovan sequence* and has a rich history. Discuss. (The On-Line Encyclopedia of Integer Sequences is a wonderful research tool for this sort of problem.)

2.3 Answers to Exercises

Answer of exercise 1. I've written the list in two columns to save space.

LLLLL
SLLLL
LSLLL
SSLLL
LLSLL
SLSLL
LSSLL
SSSLL
LLLSL
SLLSL
LSLSL
SSLSL
LLSSL
SLSSL
LSSSL
SSSSL
LLLLS
SLLLS
LSLLS
SSLLS
LLSLS
SLSLS
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SSSSS

Answer of exercise 2. Since LLSLSL has seven syllables, the list has $2^7 = 128$ meters.

Answer of exercise 3. Pingala's algorithm means that each list is twice the size of the previous one (formally, if there are k $(n-1)$-syllable meters, there are $(2 \times k)$ n-syllable meters. Since there are 2 one-syllable patterns, the numbers of meters of each length follows the pattern 2, 4, 8, 16, and so on. A formal proof would require induction.

Answer of exercise 4. Since patterns of H and T are binary, there are $2^3 = 8$ patterns, and they are HHH THH HTH TTH HHT THT HTT TTT

Each of these patterns is equally likely, and three of them have one T and two H's, so the likelihood of getting one T is 3/8 or 37.5%. In general, use the results from the study of meters, substituting H for L and T for S.

Answer of exercise 5. Start with 1, then use the list of three-syllable patterns, replacing 0 with L, 1 with S, and writing the patterns backwards. There are $2^4 = 16$ five-digit binary numbers. You write them by following this same procedure with the list of four-syllable meters. Proving that this answer is correct involves understanding how place-value number systems work in bases other than 10.

Answer of exercise 6. If your conjecture was something like "there are 13 meters with 6 beats, and you get any number in the sequence by adding the two previous," you would be correct.

Answer of exercise 7. The procedure is to add a L to all the 4-beat patterns, then add an S to all the 5-beat patterns, which are listed in figure 2.1. The answer is: LLL SSLL SLSL LSSL SSSSL SLLS LSLS SSSLS LLSS LLSLS SLSSS LSSSS SSSSSS

Answer of exercise 8. Here are the first ten entries of the Padovan sequence:

length (n)	1	2	3	4	5	6	7	8	9	10
number of patterns (P_n)	0	1	1	1	2	2	3	4	5	7

If P_n is the number of n-beat patterns, then a recursive rule is $P_n = P_{n-2} + P_{n-3}$. The proof of this statement is similar to the argument for notes of length one and two. In this case, partition the patterns of length n into patterns of length $n-2$ followed by a two-beat note and patterns of length $n-3$ followed by a three-beat note. Incidentally, a student noticed that $P_n = P_{n-1} + P_{n-5}$ and conjectured that all Padovan numbers also follow this rule. Is this correct? Hint: use the first rule to rewrite P_{n-1}.

The Padovan sequence has some beautiful properties—for example, it is related to a spiral of equilateral triangles in the way the Hemachandra-Fibonacci sequence is related to a spiral of squares, and it is closely connected to the Perrin sequence, another extremely cool sequence. See Ian Stewart's "Tales of a Neglected Number" (in *Math Hysteria*) for more examples.