Chapter 1

Prelude

The creation of mathematics and music is just two of the many ways we respond to the world around us. Mathematics describes quantity and space, while music involves the appreciation of sound. Both math and music are intentional human creations. Like mathematicians, musicians describe abstract concepts like "chords" that cannot be touched or seen. Like musicians, mathematicians speak of "beauty" or "elegance"—in their case, of a particular equation, theorem, or proof. Both disciplines rely on precise notation.

1.1 What is music?

What is the difference between "sound" and "music?" We can probably agree that the styles of popular and classical music that we make or enjoy are "music." However, it is difficult to define "music" in general. Some composers, such as John Cage (1912-1992), have deliberately challenged our notions of what music is. Although there is no universally accepted definition of music, I like the French composer Edgar Varese's description that "music is organized sound." This organization can be seen on many different levels, from the sounds that musical instruments make to the way a composer structures a piece. I would add that music exists in time and that music is a form of artistic expression. My favorite definition is

Music is the art of organizing sound in time.

Exercise 1.1. Rank these sounds in order from "most musical" to "least musical." (a) traditional Ghanaian drumming, (b) an ambulance siren, (c) raindrops on a tin roof, (d) the sound of a vacuum cleaner, (e) silence, (f) static, (g) a pop song, (h) birdsong, (i) a dog's bark, (j) freestyle rap, (k) the ticking of a clock, (i) Stockhausen's "Helicopter Quartet." Clearly, there are no right or wrong answers, but pay attention to your reasons for ranking one sound above another. Did you rate the sounds produced by humans higher? Does the presence or absence of a tone you can hum to or a rhythm you can tap to influence your decision? Which distinctions are most difficult to make? Are your judgments influenced by Western cultural norms?

Discussion questions. Throughout history, mathematicians have continually expanded the definitions of fundamental concepts like “number” and “space.” For example, zero wasn’t recognized as a number until the middle ages; the idea of four-dimensional space is relatively recent. Many composers have done the same thing. I would like to start by questioning our ideas about what music is and finding the widest possible “universe” of musical expression. In class, we watched a number of videos that challenge what’s commonly called “music.”

The Everyday Ensemble’s “Found Sound Composition.” A *found sound* is a sound that is in the world around you but not intended to be “music” by whoever or whatever made it. Some musicians record found sounds and incorporate them into their own compositions. Music can also be made with *found objects* that are not intended to be musical instruments. The Everyday Ensemble’s “Found Sound Composition” uses both found sounds and found objects. It begins with a ticking clock. The musicians play found objects—bouncing balls, rustling bags—to interlock with the rhythm of the clock. At what point did you realize that the clock is “music,” rather than, say, the beginning of a movie? How does organization play a role in our perception of music?

Ghanian drummers. In English, musical compositions are often referred to as “pieces,” implying that music comes in finite, contained units. The Ghanian drummers’ music challenges the expectation that music has a beginning and an end. In what way is time organized (or not organized) by their drumming?

John Cage’s “4’33.” Silence is a part of music—musicians even have symbols for silences, called *rests*. Can silence be music by itself? Cage focused on the listener as the creator of music. That is, the listener is the artist—the person who makes something music rather than sound. His composition 4’33” throws background sounds into the foreground. He wrote, “If music is the “enjoyment” of “sound”, then it must center on not just the side making the sound, but the side listening. In fact, really it is listening that is music. As we savor the sound of rain, music is being created within us” (from “In this time”). Do you agree? Try picking a natural sound and convince yourself that it is musical.

Algorithmic music. Algorithmic music is produced when musical notes are chosen by some mathematical set of instructions (*algorithm*). For example, the Online Encyclopedia of Integer Sequences has a “music” feature. It maps numbers to piano keys and plays the notes corresponding to numbers in a mathematical sequence. Listen to sequences A000045 (the Fibonacci numbers 1, 1, 2, 3, 5, 8, 13, …) and A005843 (the even numbers). Are these sounds “music?” What about sounds made from random sequences of numbers? (In general, music made by some random process is called *aleatory music.*) Change ringing is an algorithmic method of ringing bells that has been practiced in England for over five hundred years. Do you find it musical?

Rap music. When rap music first reached a wide audience, some people said that it “isn’t music.” Should our aesthetic judgements—what we like—have any relevance in the definition of music? What about cultural expectations? Who, if anyone, has the “right” to decide what is music?

Sonic weapons. Sonic weapons are painfully loud sounds, including tones, recorded songs, and distressing sounds such as crying babies, that are used by the military or police for crowd control or as a form of torture. They may cause permanent hearing damage. Are sonic weapons music?

Other definitions of music. Does my definition of music as "the art of organizing sound in time" include things that you do not consider "music?" Does it exclude anything that you do consider "music?" If so, give examples and state your personal definition of music.

Music notation. Some musical cultures use notation use *music notation*, which is a symbolic language used to communicate instructions to musicians. There are many different systems of notation used throughout the world. The symbols used in *Western common practice music*—the music often called "Classical"—are probably familiar to you, even if you don't know how to interpret, or "read," them. Figure 1.2 shows an example of a *musical score*. We will learn a few ways of notating music in this class.

1.2 What is sound?

Sounds are produced by rapid vibrations in the air that are detectable by the ears of humans or other animals. Musical sounds have volume (loudness or softness) and may have pitch (highness or lowness), while *rests* are silences. Both sounds and rests have a beginning (*onset*) and end (*offset*); the difference in time between their onset and offset is called their *duration*.

When you record a sound digitally, the picture of the sound that you see is called a *waveform*. It is a visual representation of the vibration of air molecules that produced the sound. The horizontal axis represents time, which is usually measured in seconds. The vertical axis shows the *displacement*—how the air molecules are moving back and forth. In reality, molecules are moving in three dimensions, but this two-dimensional picture is good enough in most situations.

Waveforms give us useful information both on a "macro" and "micro" level. On the macro, or "zoomed out," level, the rough shape of the waveform shows how the loudness of the sound changes. It often looks like a series of blobs, with fat blobs for loud sounds, since we perceive stronger vibrations as louder. If a definite onset and offset can be detected, you can find the duration of a sound by subtracting the time of onset from the time of offset. You can also zoom in on the waveform to get the "micro" picture. For many musical instruments, the zoomed-in waveform has a pattern that repeats hundreds or thousands of times per second. These patterns give information about both the pitch and the *timbre* (TAM-ber) of the sound. Timbre is difficult to describe, but, loosely, it's the quality of the sound that allows you to distinguish between different instruments and voices, even if they have the same pitch and volume.

Exercise 1.2. Install software that allows you to record and analyze sound. I recommend Audacity for a computer, TwistedWave for iPhone, and MixPad for Android. Audacity has been installed on many computers on campus. Make five-second recordings of yourself talking, humming, clapping, and saying "shh." Comment on the differences between the pattern (or lack of pattern) in their waveforms, both zoomed out and zoomed in.

Solution 1.2. Figure 1.1 shows a sample response. Each sound clip has a duration of 5 seconds. The top image shows the overall shape of the waveforms and gives you information about the rhythm of the sound. We can see that the word "music" has a duration of about a quarter second. Clapping has a repeating pattern representing the rhythm of the claps, which occur roughly at intervals of 0.4 seconds. Zooming in on the waveform gives information about the "quality" (timbre) of the sound. Only humming shows a repeating pattern. Notice that the speech transitions between a random pattern coinciding with the "s" in music and a repeating pattern coinciding with the "i." In speech, vowels look more like humming than consonants such as "s" do.

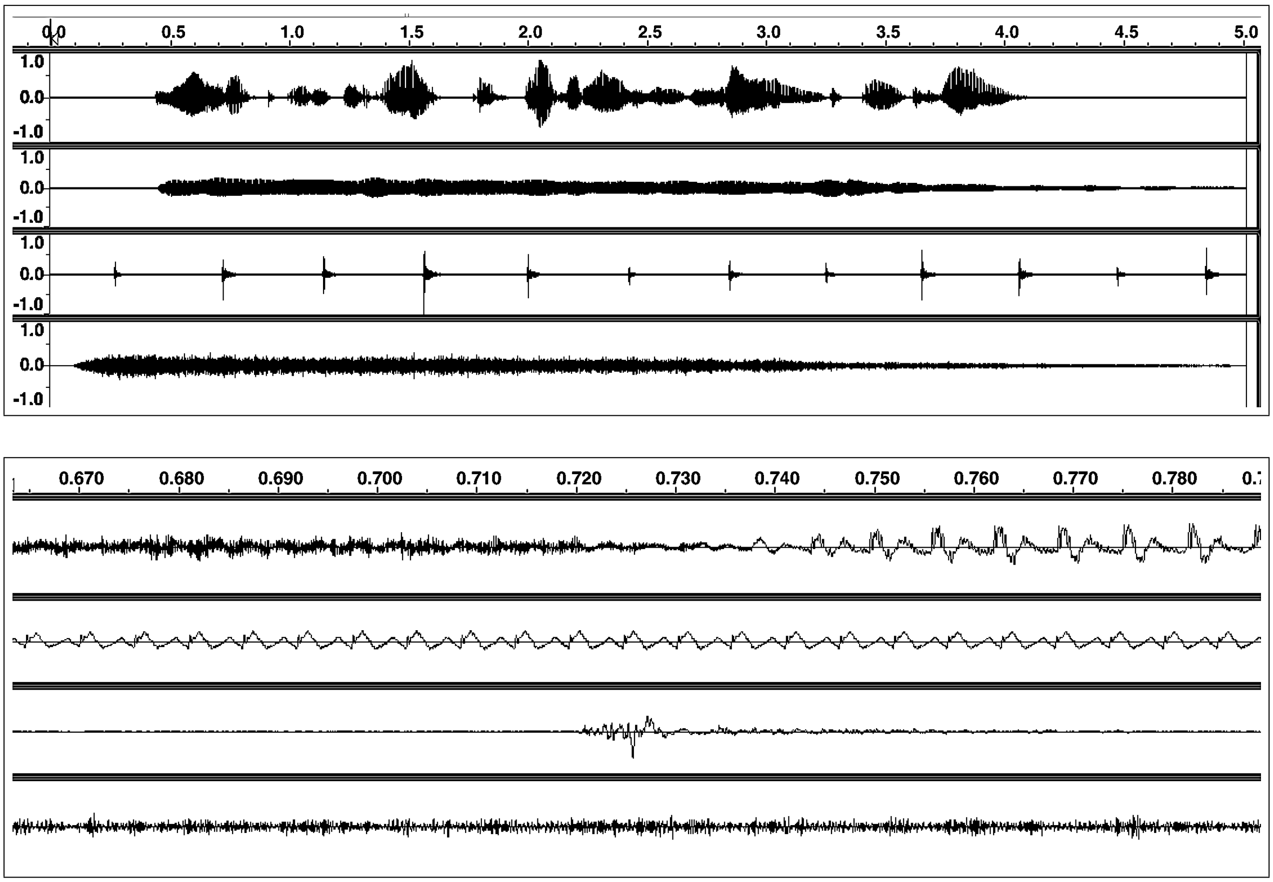

Figure 1.1: *In the top image, the first waveform is me speaking the sentence "Music is the art of organizing sound in time." The others are, in order, humming, clapping, and saying "shh." The bottom image shows what they look like around the 0.7 second mark.*

1.3 What is math?

The philosopher Michael Resnik (1981) called mathematics "a science of pattern." Mathematics precisely describes structure, both in the physical world and in the abstract. It has been part of a formal Western education for millennia. It trains us in using abstraction

Figure 1.2: *The first page of the score, or set of notated musical instructions, for Beethoven's Fifth Symphony (1808). As in the Audacity screenshot, the horizontal direction represents time, moving from left to right. Each five-line* staff *is labeled with the name of a different instrument. Symbols with oval "heads" such as* 𝅗𝅥 *and* ♩♩♩ *are notes, while the symbols* ▬, 𝄽, *and* 𝄾 *indicate rests. The position of notes in the staff indicates their pitch, with higher notes on top. The instruments play simultaneously, so that a vertical "slice" of the score shows everything that is happening at a particular moment in time. Written music like this is also called* sheet music. *We'll watch a video that synchronizes the score and music.*

and in forming logical arguments. Though few people are professional mathematicians, all humans are mathematical thinkers: we engage informally in ideas of quantity, pattern, space, and logic. Music theory, the study of structure in music, is a type of mathematical thinking.

The distinction between *mathematical science* and *mathematical thinking* is in the use of precision and rigor. Mathematical science is precise and logical. It's what you learn in math class and what mathematicians do for a living. Proofs need to be so logically rigorous that they can withstand all potential challenges. In contrast, mathematical thinking is perception or reasoning that involves number, geometry, logical reasoning, etc. Mathematical thinking doesn't have to be formal, and you don't have to have a precise answer. Reading a map and estimating a tip are examples of mathematical thinking, but not mathematical science.

Thinking about music like a mathematician: Binary codes

Some aspects of music are more mathematical than others. "Mathy" topics include anything involving patterns, structure, or numbers: music notation, rhythmic organization, music theory, the physics of sound, and music recording technology. It is difficult to use mathematics to analyze song lyrics, emotion, or expressive performance. Likewise, some kinds of music use mathematics more explicitly than others. Examples of "mathy" music include algorithmic music, serial music, spectral music, and math rock. All of these genres of music derive some of their structure from mathematics. Unsurprisingly, mathematical thinking about music has focused on certain topics and kinds of music and mostly ignored others.

As an illustration of the type of reasoning mathematicians use, let's start with the theory of binary codes. Although developed by mathematicians, this theory has applications in music, computer engineering, statistics, and more. A *binary code* is a pattern of any length formed by two symbols. Examples of binary codes include "`011001`," "`x--x-xx--xx-x`", and "𝅗𝅥 ♩ ♩ 𝅗𝅥 𝅗𝅥." Each of the two symbols may appear any number of times (including zero times) in a code. The length of a code is the number of symbols it contains, so that the length of "`011001`" is six. The order of the symbols makes a difference. For example, `001101` and `111000` are different codes.

Binary codes show up in music in several ways. *Drum tablature* is music notation formed by two symbols: "`x`" represents a strike of the drum and "`-`" represents a rest. Each `x` or `-` takes the same amount of time, which I'll call a "beat." An example of an eight-beat drum pattern is `x-x-x---`.

Binary codes are intrinsic to digital sound recording, and, in fact, all digital information is stored as binary codes. Data on a CD are stored as "pits" and "lands" that can be read by the laser beam in a CD player. Each symbol in the code is called a "bit." Every second of a CD is encoded by 1,411,200 bits, so a 75-minute CD requires $1,411,200 \times 60 \times 75 = 6350400000$ bits of storage. Since there are eight million bits per megabyte (MB), this information requires 793.8 MB of storage.

While a musician might ask,

Which drum pattern should I play?
How long can my CD be?

mathematicians study the theoretical possibilities of binary codes. For example, a mathematician might ask,

How many binary codes of a given length are possible?
How can I write all the codes of a given length?
Do any binary codes have interesting patterns?
What about "ternary codes," using three symbols?
What about codes using any fixed number of symbols?

In asking questions about binary codes, mathematicians use *abstraction*—that is, they remove the problem from its original context and study the concept of "binary code" in general. An advantage of abstraction is that mathematicians are able to study all kinds of binary codes at once, rather than having to redo the problem every time it appears in a different real-world situation.

Mathematical solutions to a problem may inspire a composer. For example, writing all possible patterns could lead to the discovery of interesting patterns that are not normally used in music. The fact that many melodies are patterns of seven notes means that understanding codes with seven symbols (rather than two) many help us understand melodic possibilities.

The number of binary codes. The length of a binary code is the number of symbols in that particular code. For example, `x--x-x-` has length 7. How many binary codes of a given length are possible? Here's a chart, using the symbols 0 and 1. This evidence suggests that there are 2^n binary codes with length n.

Length	Codes	Num. of codes
1	0, 1	$2 = 2^1$
2	00, 01, 10, 11	$4 = 2^2$
3	000, 001, 010, 011, 100, 101, 110, 111	$8 = 2^3$
4	0000, 0001, 0010, 0011, 0100, 0101, 0110, 0111, 1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111	$16 = 2^4$

Table 1.1: *Counting binary codes of length 1 through 4.*

Listing binary codes. A set of instructions—an *algorithm*—for writing all the possible binary codes of a given length is more useful to a musician than a formula. To write the codes of length $n + 1$, write the codes of length n twice, then add 0's to the beginning of the first group of codes and 1's to the second group of codes. This algorithm explains the 2^n formula for the number of binary codes: the number of possible codes doubles when you increase the length by one.

Exercise 1.3. Find the number of eight-beat drum patterns formed of strikes (`x`) and rests (`-`).

Exercise 1.4. Write all the four-beat patterns in drum tab. Clap through the rhythms, repeating each rhythm four times. Which are easiest to play? hardest? Do you find any of the rhythms particularly interesting?

Exercise 1.5. A 75-minute CD is a binary code of length approximately 6,350,400,000 bits. How many CDs are possible?

Patterns of short and long notes. Here's another musical problem that turns out to have a surprisingly mathematical answer. Suppose you want to fill the space of four beats with some combination of short notes that last one beat (write as "1" or ♩) and long notes that last two beats (write as "2" or 𝅗𝅥) For example, 112 represents two one-beat notes followed by a two-beat note, for a total duration of four beats. You could also write the pattern 112 as ♩♩𝅗𝅥. How many different ways can you fill four beats with one-beat notes and two-beat notes? The answer is five: 1111, 112, 121, 211, and 22. In music notation, that's ♩♩♩♩, ♩♩𝅗𝅥, ♩𝅗𝅥♩, 𝅗𝅥♩♩, and 𝅗𝅥𝅗𝅥. As in the previous problem, make a table of patterns. This time, list patterns that have the same duration, which is found by adding the numbers in the pattern (for example, the duration of 11212 is $1 + 1 + 2 + 1 + 2 = 7$).

Duration	Patterns	Num. of patterns
1	1	1
2	11, 2	2
3	12, 111, 21	3
4	112, 22, 121, 1111, 211	5
5	122, 1112, 212, 1121, 221, 1211, 11111, 2111	8

Table 1.2: *Counting rhythm patterns of duration 1 through 5.*

The sequence of numbers of patterns is 1, 2, 3, 5, 8, which you might recognize as the beginning of the Fibonacci sequence

$$1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$$

Each number in the sequence is the sum of the previous two numbers. This fact was discovered by the Indian scholar Hemachandra around 1150 AD while studying rhythm in poetry. In India, the numbers are known as the *Hemachandra numbers*.

Hemachandra's problem is equivalent to the "domino-square problem": in how many ways can 1×2 dominoes and 1×1 squares tile a $1 \times n$ rectangle? Figure 1.3 gives a visual proof

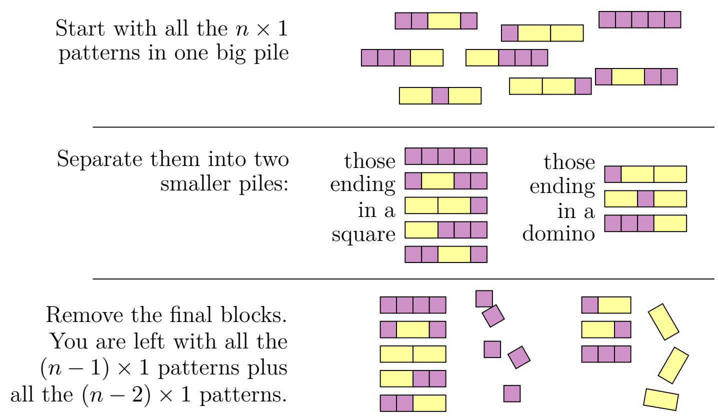

Figure 1.3: *Here is a visual demonstration that the nth number in the sequence of numbers of domino-square tilings is the sum of the two preceding numbers.*

each number in the sequence is the sum of the previous two numbers, and therefore that the answer is the nth Fibonacci number.

Exercise 1.6. Find the number of rhythm patterns formed of notes of length 1 or 2 that have duration equal to 8 beats.

Exercise 1.7. Find the number of rhythm patterns formed of notes of length 1 or 2 that have duration less than 8 beats.

Exercise 1.8. Find the number of rhythm patterns formed of notes of length 1 or 2 that have duration equal to 8 beats and start with "1". Find the number that start with "2." How is this question related to Fibonacci numbers?

Exercise 1.9. Explain how the "rhythm pattern" problem and the domino-square problem are related and why they have the same solution.

Mathematical terminology

Here are some of the basic terms that mathematicians use to describe what they do.

Number.

- The *natural numbers* are the numbers $1, 2, 3, 4, \ldots$.
- The *integers* are the numbers $\ldots -3, -2, -1, 0, 1, 2, 3, \ldots$.
- The *real numbers* are all the numbers on the number line, including numbers like $-\frac{1}{3}$, 0.278, $\frac{22}{7}$, π, and $-0.\overline{7}$.

Definition. A *definition* is a precise description of a mathematical term. Here are a few useful definitions that you should know:

- For integers a and b, we say that a *divides* b if a is nonzero and b equals a times some integer k (that is, $b = ak$).
- An *even number* is a number that is divisible by 2. In other words, it equals $2k$, where k is an integer.

Proposition. A *proposition* is a statement that is either true or false. "Two is an even number" and "$3 + 2 = 5$" are true propositions. "Two is an odd number" is a false proposition. "Figure out whether two is even," "$5 + x = 2$," and "Is two an even number?" are not propositions. "This sentence is false" sounds like it might be a proposition, but it's neither true nor false. It is an example of a *paradox*.

Axiom. In order to do any useful mathematics, we all have to agree that certain statements, called *axioms*, are true without proof. For example, the *Peano axioms* are needed to establish the definition of integers and the rules of arithmetic. The *Parallel postulate* in geometry states that parallel lines never intersect, where two lines are *parallel* if they are perpendicular to the same line. Surprisingly, it is not possible to use the basic definitions of geometry to prove that parallel lines do not intersect. In fact, the parallel postulate must be accepted as true in Euclidean geometry, which is the geometry of the plane that you learned in high school.[1]

Theorem. A *theorem* is a proposition pertaining to mathematics or logic that has been proven to be true.[2]

Proof. A *proof* is a rigorous, logical argument, written in complete sentences, that demonstrates that a theorem or proposition is true. Proofs build on definitions, axioms, demonstrably true propositions, and other theorems.

The proposition "the sum of two even numbers is an even number" can be proven using a logical argument, assuming that the normal rules of arithmetic are true and following from the definition of an even number. Here is the proof. The box at the end of the paragraph marks the end of the proof.

> Proposition: The sum of any two even numbers is an even number.
>
> *Proof.* Suppose x and y are even numbers. The definition of "even number" means x and y are divisible by 2, so there are integers k and l where $x = 2k$ and $y = 2l$. By the rules of addition and multiplication, $x + y = 2k + 2l = 2(k + l)$, which is an even number because $k + l$ is an integer. □

[1] Parallel lines *do* intersect on the globe, and this type of geometry is called *spherical geometry*.

[2] The distinction between "theorems" and "true propositions" is not hard-and-fast. Theorems are typically more important, useful, universal, or just more difficult to prove than ordinary true propositions.

Notice that this proof implicitly relies on the Peano axioms, because we need the rules of arithmetic and the definition of integers to complete the proof.

Conjecture. A *conjecture* is a proposition pertaining to mathematics or logic that is made without proof. Conjectures are usually "educated guesses" based on some pattern that has been observed. "The sum of two integers is greater than either integer" is a conjecture. Although this conjecture is easily disproven, there exist some famous conjectures that mathematicians have never been able to prove true or false.

Example. An *example* is an illustration of a proposition in a particular instance. Unless you are able to demonstrate that all possible examples of a proposition are true, examples do NOT prove the proposition.

Universal proposition. A *universal proposition* is one that is asserted for infinitely many examples. The universal proposition "the sum of any two even numbers is an even number" is illustrated by examples such as $2 + 4 = 6$ and $-4 + 4 = 0$. However, examples can't prove that this (or any) universal proposition is true, because there are infinitely many examples, and you can't test them all.

Counterexample. A *counterexample* is an example that proves that a proposition is false. A counterexample to "The sum of two integers is greater than either integer" is $-1 + 2 = 1$, which is not larger than 2.

Algorithm. An *algorithm* is a set of mathematical instructions. An algorithm for determining whether a number is even is the following: Divide the number by 2. If the remainder is 0, the number is even. If the remainder is not 0, the number is not even.

Exercise 1.10. Prove that 0 is an even number.

Exercise 1.11. A number is *odd* if it equals $2n + 1$, where n is an integer. Suppose that a student wishes to prove that the sum of any two odd numbers is an even number. Explain why the following is not a proof.

> Proposition: The sum of any two odd numbers is an even number.
>
> *Proof.* We can test odd numbers: $1 + 1 = 2$, $1 + 3 = 4$, $3 + 1 = 4$, $1 + 5 = 6$, etc. This also works for negative odd numbers. For example, $-1 + 3 = 2$, $-1 + (-3) = -4$, and $-5 + 5 = 0$. In each case, odd numbers sum to an even number. $\square$

Exercise 1.12. Find a counterexample to the proposition "the product of any two odd numbers is an even number."

Exercise 1.13. Explain why the proposition "All music is made by musical instruments or voices" is false.

Exercise 1.14. Explain why many examples don't prove that a universal proposition is true, but one counterexample *disproves* a universal proposition.

1.4 Solutions to exercises

Solution 1.3. There are $2^8 = 256$ patterns, because eight-beat patterns correspond to binary codes with length eight.

Solution 1.4. There are $2^4 = 16$ patterns. We can use the algorithm to list them, replacing "0" with `x` and "1" with `-`. The patterns are

```
xxxx    -xxx
xxx-    -xx-
xx-x    -x-x
xx--    -x--
x-xx    --xx
x-x-    --x-
x--x    ---x
x---    ----
```

Notice the last rhythm—the John Cage rhythm! The rhythms that begin with a strike are typically easier to clap.

Solution 1.5. There are $2^{6350400000}$ possible CDs, a number with more than a billion digits!

Solution 1.6. The answer is the eighth Fibonacci number, which is 34.

Solution 1.7. Add the numbers of patterns with duration 1 through 7 beats to get $1 + 2 + 3 + 5 + 8 + 13 + 21 = 53$.

Solution 1.8. Every pattern with 8 beats that starts with "1" is a "1" followed by a pattern with duration 7. There are 21 patterns with duration 7. Every pattern with 8 beats that starts with "2" is a "2" followed by a pattern with duration 6. There are 13 patterns with duration 6. Notice that $21 + 13 = 34$. In general, every pattern with duration n is either a "1" followed by a pattern with duration $n - 1$ or a "2" followed by a pattern with duration $n - 2$. This proves that the number of patterns of duration n is the sum of the number of patterns with duration $n - 1$ and the patterns with duration $n - 2$. That is, the Fibonacci relationship is true for all $n > 2$.

Solution 1.9. A rhythm of duration n corresponds to a $n \times 1$ rectangle, a note of duration 1 corresponds to a 1×1 square, and a note of duration 2 corresponds to a 2×1 domino. Every pattern of 1's and 2's corresponds to an order of squares and dominoes, and vice versa.

Solution 1.10. *Proof.* Since $0 = 2 \cdot 0$ and 0 is an integer, 0 is even by the definition of even. □

Solution 1.11. The student has *not* tested all odd numbers. That would be impossible because there are infinitely many odd numbers. A logical argument must be used, such as the proof on page 1–10.

Solution 1.12. A counterexample is $5 \cdot 3 = 15$, because 5 and 3 are odd and 15 is also odd.

Solution 1.13. You need to find a counterexample—that is, some music that is not made by musical instruments or voices. Two counterexamples are music using found sounds and algorithmic music made by a computer.

Solution 1.14. In the case of a true universal proposition, there are infinitely many examples to test, and testing some of them does not prove that they are all true. Therefore, the rules of logic must be used. In the case of a false universal proposition, demonstrating that it is false in one instance means that the proposition is *not* universal and therefore is false.