Chapter 4

Interval, Pitch, and Scale

The science of musical sound can only take us so far along the road to appreciating, analyzing, and composing music. Intervals, or relationships between pitched sounds, are almost more important that the sounds themselves. They form the underpinnings of melody and harmony. In this chapter, we will investigate the role of intervals in our understanding of music. This leads to the definition of pitch—a measure of frequency that corresponds to human perception—and pitch class, which models our hearing that pitched sounds an octave apart are fundamentally similar.

4.1 Intervals

Though stringed instruments and most wind instruments have different sounds and fundamental frequencies, the sequence of ratios between their harmonics and fundamental is the same. In general, when comparing two frequencies, **their ratio is the quantity that is musically relevant**.

Definition 4.1 (Frequency definition of interval)**.** *An* interval *is a relationship between two frequencies that is determined by their ratio. Two pairs of pitched sounds form the same interval if and only if their ratios are either equal or reciprocals.*

For example, the interval formed by 120 Hz and 360 Hz is the same as the interval formed by 900 Hz and 300 Hz, because both pairs of frequencies, with the higher listed first, are in a 3:1 ratio[1]

The most fundamental musical interval is the *octave*, which corresponds to a 2:1 frequency ratio. The first and second harmonics of a string form an octave because $\frac{2f}{f} = \frac{2}{1}$, where f is the fundamental frequency. No matter at what frequency you start singing, the ratio

[1]Remember that ratios are a way of writing fractions, so the fact that $\frac{360}{120} = \frac{900}{300} = \frac{3}{1}$ proves that the intervals are equal. Ratios should be reduced to lowest terms. For ease of comparison, I'll normally write them with the larger number first; however, ratios that are reciprocals, like 2:3 and 3:2, represent the same interval.

between the frequencies of the first two notes in "Somewhere, Over the Rainbow" is an octave.

Since the harmonics of a string, f, $2f$, $3f$, $4f$, ..., are each multiples of the fundamental f, the ratios formed by consecutive harmonics are 2:1, 3:2, 4:3, 5:4, and 6:5. These intervals all have names that are familiar to musicians: 2:1 corresponds to an octave, 3:2 to a perfect fifth, 4:3 to a perfect fourth, 5:4 to a major third, and 6:5 to a minor third, as summarized in Figure 4.1. A perfect fifth is the interval between the second and third notes of "Twinkle, twinkle, little star," a fourth is the interval between the first two notes of "Amazing grace," a major third is the interval between the first two notes of "When the saints go marching in," and a minor third is the interval between the first two notes of "Greensleeves." Again, the frequency at which you start singing doesn't matter—just the interval between the first two frequencies.

2:1	3:2	4:3	5:4	6:5
octave	fifth	fourth	major third	minor third
Somewhere Over the Rainbow	Twinkle, Twinkle	Amazing Grace	When the Saints	Greensleeves

Figure 4.1: *Intervals between the first six harmonics and songs to remember them by.*

Exercise 4.1. Determine which pairs of frequencies form the same interval:
(a) 600 Hz and 400 Hz, (b) 600 Hz and 500 Hz, and (c) 600 Hz and 900 Hz. Listen to each interval by generating two simultaneous tones in Audacity. Which intervals sound most similar? What if you use the "Solo" option to listen to the two tones in an interval separately, rather than simultaneously?

Exercise 4.2. Find a pair of frequencies that forms the same interval that 300 Hz and 400 Hz do. Find the *other* frequency that forms a 4:3 ratio with 300 Hz. Find the formulas for all the frequencies that form a 4:3 ratio with a frequency f.

Exercise 4.3. Use Figure 4.1 to find the names of the intervals between (a) the third and sixth harmonics of a string and (b) the fourth and sixth harmonics of a string.

4.2 Consonant intervals

Although any pair of frequencies form an interval, not every interval is used in music equally often. The intervals that are most important for constructing music are called consonant intervals. Their sound is more "pleasant" or "smooth" than any other intervals for several reasons. Consonant intervals are so fundamental to music that musicians have special names for them, and they are the basis of constructing both melody and harmony.

Let's start by making a demonstration of all possible intervals whose ratio falls between 1:1 and 4:1. I chose those limits because it turns out that understanding which of these intervals are consonant predicts which larger or smaller ratios are consonant. Generate

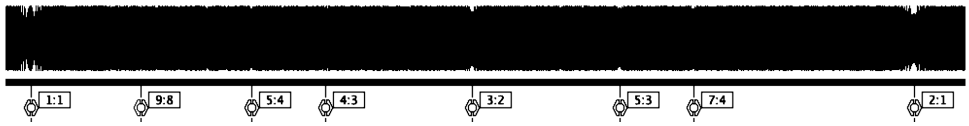

Figure 4.2: *Consonant intervals correspond to momentary smoothness of the envelope of the sound wave created by mixing a constant tone with a rising tone. These points look like "notches" in the waveform when you zoom out. The ratios shown represent the ratios between the frequencies of the tones.*

two simultaneous 20 minute tracks: a 220 Hz sawtooth wave and a sawtooth wave that interpolates linearly between 200 Hz and 1400 Hz, then mix them into one track. Magnify the wave and locate times at which the mixed track becomes momentarily "smooth"—we will call these points consonances. Note that the envelope of the waveform looks roughest when it is close, but not equal to, a consonance.

Noting that the second tone ascends by 1 Hz per second, you can calculate the frequency of the ascending tone at each consonance. The most noticeable consonance occurs when the ascending tone reaches 220 Hz—that is, when the ratio formed by the two tones is 1:1. The 2:1 ratios and 4:1 ratios are also easy to spot. Starting at 200, some consonances I found occur when the ascending tone reaches, in Hertz, 220, 293, 330, 352, 440, 528, 550, 587, 660, 733, and 880, which all form ratios with 220 that are equal or approximately equal to small integer ratios. For example, 330:220 equals 3:2 and 293:220 is approximately equal to 4:3. Figure 4.2 shows the mixed track in the area between the 1:1 and 2:1 ratios. When you play the track near these points, you hear consonant intervals that may sound familiar. Now turn back to Figure 4.1, which shows the intervals between harmonics. *All* of the intervals that are formed by relationships between the first six harmonics of a wind or stringed instrument are consonant.

Why do consonant intervals sound special, and why do intervals that are near consonances sound especially "rough," or dissonant? Near the point where the two tones are in 1:1 ratio, or unison, the combination of the tones causes beating at a rate equal to the difference in frequencies. If the difference is very small, you hear a slow pulsation that is not unpleasant. However, when the difference is around 20 Hz, the beating effect is especially grating. A related effect happens near every consonant interval. When two tones are vibrating in an exact ratio such as 2:1 or 3:2, the combination waveform repeats in a predictable pattern and sounds "smooth."

4.3 Measuring intervals using semitones

Our ears don't judge musical "distance" between two notes (pitched sounds) in the same way as we measure frequency. Suppose you start singing the song "Somewhere Over the Rainbow" at 220 Hz. Since the relationship between the first and second notes of the song is an octave, the second note's frequency is 440 Hz. If you start singing at 330 Hz, then the second note of the song is 660 Hz. Both pairs of frequencies are exactly the same distance apart on the piano—twelve keys. The relevant computation is the 2:1 octave relationship, not the differences in frequencies, which are 220 Hz and 330 Hz, respectively.

Definition 4.2. *A* semitone (ST) *is a unit of measuring musical intervals. One octave equals 12 semitones. Two intervals are equal if and only if they equal the same number of semitones. There are 100* cents *per* semitone.

Semitones are a unit of measurement that corresponds to our perception about musical "distance." There is one semitone between the frequencies of adjacent[2] keys on the piano. You can hear a semitone prominently in the soundtrack from the movie "Jaws."

Semitones are supposed to "measure" intervals, which are determined by ratios, not distances. Although we normally use subtraction to measure distances, ratios are formed by division. For example, the distance in semitones between 440 Hz and 880 Hz is the same as the distance between 330 Hz and 660 Hz, because both pairs form a 2:1 ratio—an octave. In these examples, the distance is 12 ST. However, even though $660 - 330 = 330$ and $990 - 660 = 330$, the semitonal distance between 330 Hz and 660 Hz is *not* the same as the distance between 660 Hz and 990 Hz, because $\frac{660}{330} \neq \frac{990}{660}$.

What is the mathematically correct way to "measure" a ratio? Let's start with octaves. We know that the 2:1 ratio should measure 12 ST. In the table, the variable r is the fractional version of a ratio, so that the ratio $4:1$ corresponds to $r = \frac{4}{1} = 4$.

octaves	ratio	r	ST
1	$2:1$	$2 = 2^1$	$12 = 1 \cdot 12$
2	$4:1$	$4 = 2^2$	$24 = 2 \cdot 12$
3	$8:1$	$8 = 2^3$	$36 = 3 \cdot 12$

For intervals that are not octaves, we need the logarithm base 2 ($\log_2$), which is the mathematical function that inputs 2^n and outputs its exponent n (that is, $\log_2 2^n = n$). For example, $\log_2(8) = \log_2(2^3) = 3$ and $\log_2(\frac{1}{2}) = -1$ because $2^{-1} = \frac{1}{2}$. A number doesn't have to be an integer power of 2 for its logarithm to be computed: $\log_2(7) \approx 2.807$ because $2^{2.807} \approx 7$. Most calculators won't compute $\log_2$ directly, but you can use the formula $\log_2(a) = \log(a)/\log(2)$. For example, $\log_2(7) = \log(7)/\log(2)$.

Measuring intervals in semitones. The measure, s, in ST, of the interval with frequency ratio $A:B$ is

$$\boxed{s = 12\log_2(r) = 12\log(r)/\log(2) \text{ ST, where } r = A/B.}$$

Example. An octave is a 2:1 ratio, so $r = 2$ and $s = 12\log(2)/\log(2) = 12$ ST. This confirms something we already know: there are twelve semitones per octave. The frequency ratio between the second and third harmonics of a string is 3:2. It corresponds to a difference in pitch of $12\log(3/2)/\log(2) \approx 7.020$ ST.

[2]"Adjacent" is somewhat ambiguous, since the black keys are shorter than the white ones. I'll say that keys are adjacent if they are next to each other on the ends farthest from the pianist's fingers, or as if the black keys extended out to the edge of the keyboard, as in Figure 4.3.

Converting from semitones to ratios. Suppose you are given a distance, s, in semitones and want to know what frequency ratio it represents. Solving the previous equation for r shows that the ratio corresponding to s ST is

$$\boxed{\text{ratio} = r : 1, \text{ where } r = 2^{s/12}}$$

Example. There are 12 semitones in an octave; substituting $s = 12$ in the formula gives us $r = 2^{12/12} = 2$, confirming that the frequency ratio for an octave is 2:1. We can also convert 7 semitones into a ratio: $r = 2^{7/12} \approx 1.498$, so the ratio is $1.498 : 1$, or approximately $3 : 2$.

Exercise 4.4. We showed that a 2:1 ratio of frequencies corresponds to exactly 12 ST and a 3:2 ratio corresponds to approximately 7.020 ST. Find the measure, in semitones, of the other ratios formed by the first six harmonics: 4:3, 5:4, and 6:5. Round your answers to two decimal places.

Exercise 4.5. Find ratios approximating the distances of (a) 5 ST, (b) 4 ST, and (c) 3 ST. Round your answers to two decimal places.

4.4 Absolute pitch

So far, I have defined the semitone as a unit of measurement, like an inch or a liter. In circumstances such as tuning a piano, you would want to have an "absolute" way of locating frequencies so that the distances between them correspond to semitones, in accord with human perception. Any logarithmic measure of frequency is called *pitch*. There are several different systems in use. We'll be using MIDI note numbers, in which each key on the piano corresponds to a whole number from 21 to 108. MIDI stands for Musical Instrument Digital Interface and is an industry standard for electronic instruments.

Definition 4.3. MIDI note numbers *are a system of indicating pitch. They are assigned so that note 69 corresponds to 440 Hz and an interval of one semitone corresponds to a difference of 1 in MIDI note number.*

Figure 4.3 shows a piano keyboard with the note names, MIDI note numbers, and frequencies labeled. Two important keys are highlighted: key 69 (A440) and key 60 (*middle C*). I'll use the words "pitch" and "MIDI note number" to mean the same thing.

To find the frequency of pitch p, measure its distance from note 69 in semitones, convert that measurement to a frequency ratio, and find the frequency that forms the desired ratio with 440 Hz. For example, let's calculate the frequency for note 60 (middle C). The distance between 60 and 69 is 9 ST. Using the semitones-to-ratio conversion, $r = 2^{9/12}$, which tells us that the ratio between 440 Hz and the frequency of middle C is $2^{3/4} : 1$. To find the answer, divide 440 by $2^{3/4}$ to obtain approximately 261.626 Hz. Figure 4.3 shows that this is correct.

note	MIDI	frequency
C	108	4186.009
B	107	3951.066
A#/Bb	106	3729.310
A	105	3520.000
G#/Ab	104	3322.438
G	103	3135.963
F#/Gb	102	2959.955
F	101	2793.826
E	100	2637.020
D#/Eb	99	2489.016
D	98	2349.318
C#/Db	97	2217.461
C	96	2093.005
B	95	1975.533
A#/Bb	94	1864.655
A	93	1760.000
G#/Ab	92	1661.219
G	91	1567.982
F#/Gb	90	1479.978
F	89	1396.913
E	88	1318.510
D#/Eb	87	1244.508
D	86	1174.659
C#/Db	85	1108.731
C	84	1046.502
B	83	987.767
A#/Bb	82	932.328
A	81	880.000
G#/Ab	80	830.609
G	79	783.991
F#/Gb	78	739.989
F	77	698.456
E	76	659.255
D#/Eb	75	622.254
D	74	587.330
C#/Db	73	554.365
C	72	523.251
B	71	493.883
A#/Bb	70	466.164
A	69	440.000
G#/Ab	68	415.305
G	67	391.995
F#/Gb	66	369.994
F	65	349.228
E	64	329.628
D#/Eb	63	311.127
D	62	293.665
C#/Db	61	277.183
middle C	60	261.626
B	59	246.942
A#/Bb	58	233.082
A	57	220.000
G#/Ab	56	207.652
G	55	195.998
F#/Gb	54	184.997
F	53	174.614
E	52	164.814
D#/Eb	51	155.563
D	50	146.832
C#/Db	49	138.591
C	48	130.813
B	47	123.471
A#/Bb	46	116.541
A	45	110.000
G#/Ab	44	103.826
G	43	97.999
F#/Gb	42	92.499
F	41	87.307
E	40	82.407
D#/Eb	39	77.782
D	38	73.416
C#/Db	37	69.296
C	36	65.406
B	35	61.735
A#/Bb	34	58.270
A	33	55.000
G#/Ab	32	51.913
G	31	48.999
F#/Gb	30	46.249
F	29	43.654
E	28	41.203
D#/Eb	27	38.891
D	26	36.708
C#/Db	25	34.648
C	24	32.703
B	23	30.868
A#/Bb	22	29.135
A	21	27.500

Figure 4.3: *Note names, MIDI note numbers (pitches), and frequencies for the piano keyboard.*

This computation is summarized by the pitch-to-frequency conversion formula. The frequency corresponding to pitch p is

$$\boxed{f = 440 \cdot 2^{(p-69)/12} \text{ Hz.}}$$

The frequency-to-pitch conversion formula is found by solving the above formula for p. The pitch corresponding to frequency f is

$$\boxed{p = 69 + 12\left(\frac{\log(f/440)}{\log(2)}\right).}$$

Pitches don't have to be whole numbers. Non-integer pitches are called *microtones*—they fall "in the cracks" between piano keys.

Exercise 4.6. Choose any key on the keyboard in Figure 4.3 and use the pitch-to-frequency formula to check that the frequency listed is correct.

Exercise 4.7. Find two pairs of pitches that are six semitones apart. Look up their frequencies in Figure 4.3 and compute the frequency ratio for each pair.

Exercise 4.8. List the six frequencies that are one, two, or three octaves above or below 440 Hz. In Figure 4.3, mark all the piano keys corresponding to these frequencies. Find the pitches of the piano keys you marked.

Exercise 4.9. Approximate to two decimal places (a) the pitch of 660 Hz; (b) the frequency of pitch 1; (c) the frequency of pitch 13; (d) the pitch whose frequency is in a 9:4 ratio above 440 Hz

Exercise 4.10. List the pitches of all the keys named "C" on the keyboard, from smallest to largest. How are these numbers related mathematically? List the numbers of all the "G"s. How are these numbers related?